HEAT TRANSFER

MODULE 1

INTRODUCTION AND 1D STEADY STATE HEAT CONDUCTION

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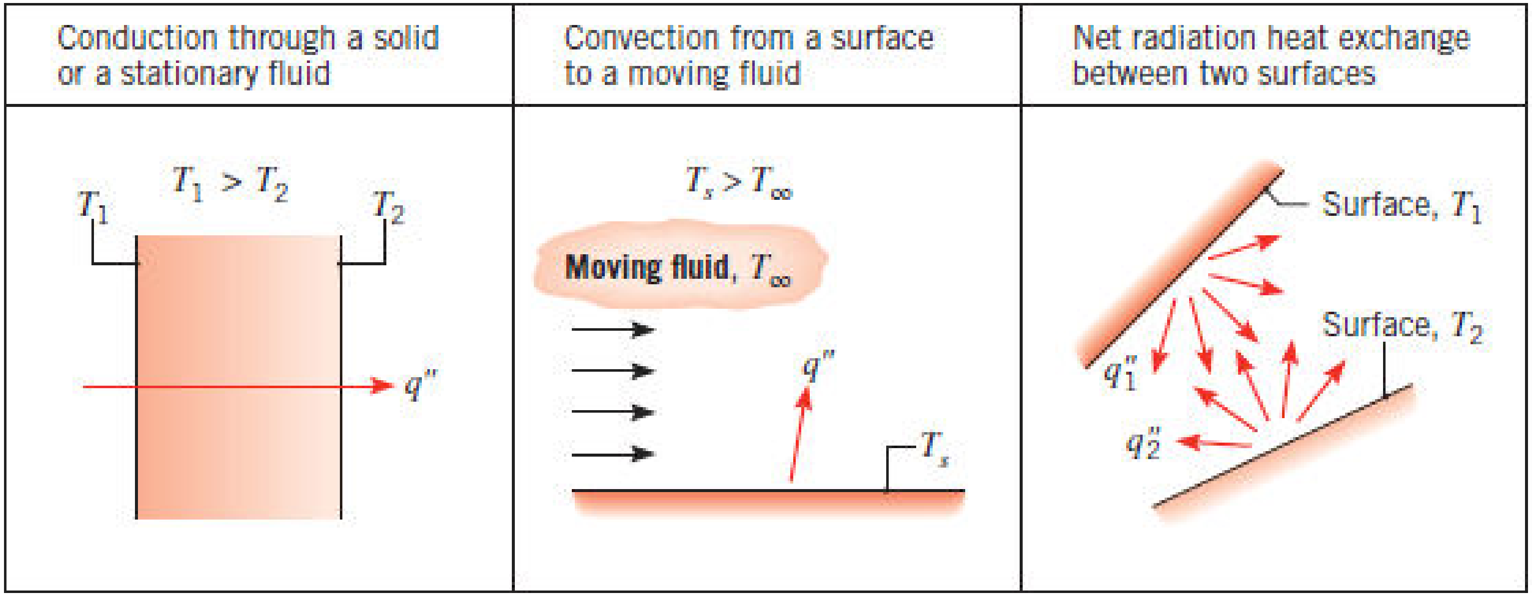
INTRODUCTION - MODES AND MECHANISMS OF HEAT TRANSFER

- Thermodynamics deals with heat and work interactions
- Thermodynamics deals with end states of process interaction and no information on nature of interaction or the time rate
- Heat transfer supplements thermodynamic analysis by probing modes of heat transfer. It deals with numerical relations of heat transfer rates
- Heat transfer knowledge in conjunction with first law of thermodynamics aid in solution to technological problems

What is heat transfer and how is heat transferred?

- Heat transfer or heat is thermal energy in transit due to spatial temperature difference
- Temperature gradient in a solid medium or fluid results in heat transfer by **conduction**
- Heat transfer between a surface and a moving fluid that are at different temperatures is called **convection**
- All surfaces of finite temperature emit heat in the form of electromagnetic waves . Such net heat transfer is called thermal **radiation**



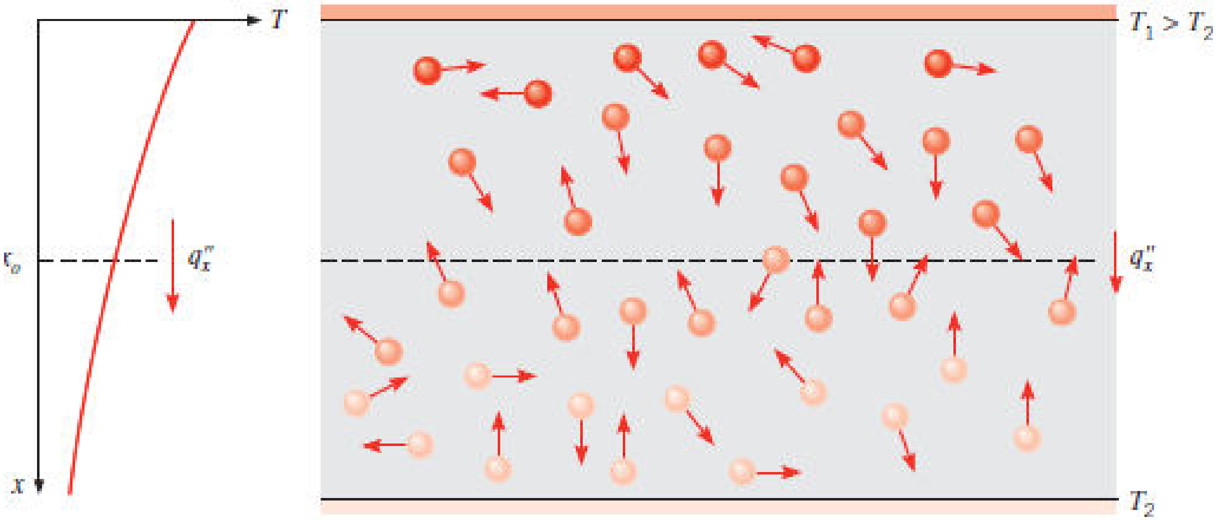
Conduction through a solid or a stationary fluid

Convection from a surface to a moving fluid

Net radiation heat exchange between two surfaces

Physical origin of rate equation - Conduction

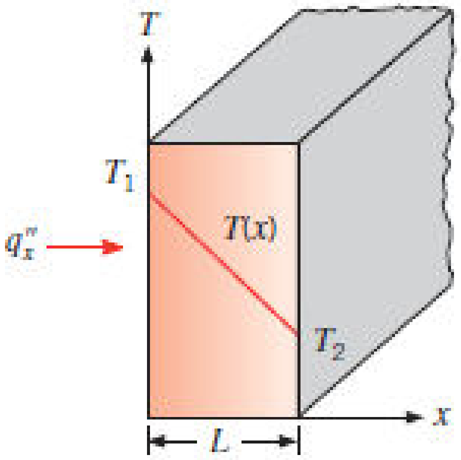
- Conduction is the transfer of energy from more energetic particle to less energetic particle due to their mutual interaction
- Temperature at a point in medium corresponds to random translation motion, rotation and vibrational motion of molecules
- Energy transfer occurs either by collision of molecules or by random motion in the presence of temperature gradients
- In solids, thermal energy transfer occurs due to lattice vibrations and translational motion of free electrons



Heat rate equation for conduction – Fourier's law

- For one dimensional plane wall having temperature distribution T(x) , the rate equation is expressed as:

$$q_x'' = -k\frac{dT}{dx}$$



- Heat flux q" (W/m^2) in x-direction perpendicular to direction of heat transfer is proportional to dT/dx
- Parameter k, is a transport property known as thermal conductivity (W/m.K) is characteristic of wall material
- Minus sign is a consequence of fact that heat is transferred in the direction of decreasing temperature
- Under steady state conditions, temperature distribution is linear and the temperature gradient becomes: $\frac{dT}{dx} = \frac{T_2 - T_1}{L}$

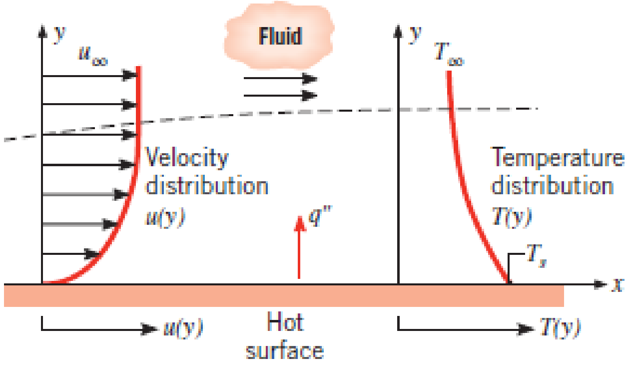
$$q_x'' = -k\frac{T_2 - T_1}{L}$$

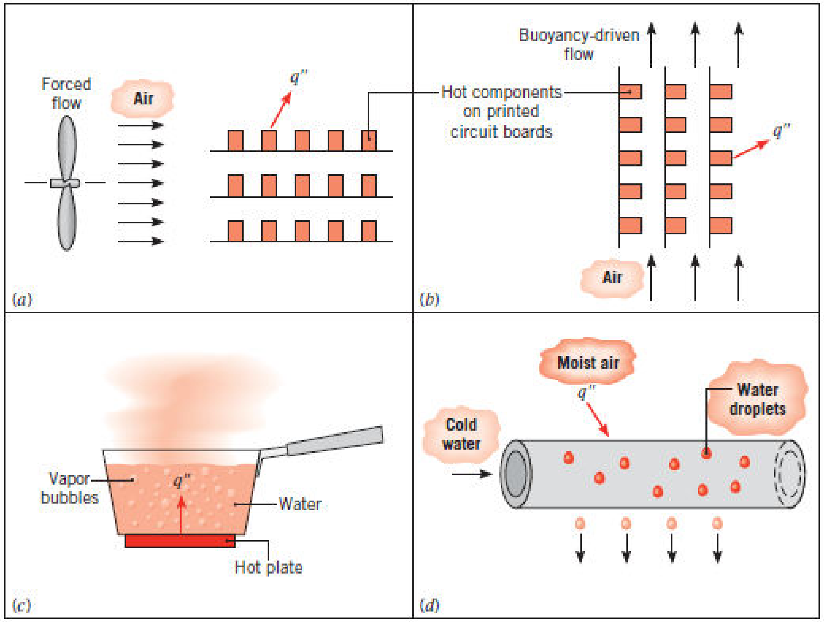
$$\boxed{q_x'' = k\frac{T_1 - T_2}{L} = k\frac{\Delta T}{L}}$$

- Heat rate by conduction , q_x (W) through a plane wall of area A is then q_x'' . A

Heat rate equation for convection – Newton’s law of cooling

- Convection heat transfer mode comprise of two mechanisms: random molecular motion (diffusion) and bulk/macroscopic motion of fluid
- Convection is referred to cumulative transport and advection refers to bulk fluid motion alone
- <u>Fluid - surface interaction results in the formation of hydrodynamic boundary layer</u>
- In case of thermal gradient between surface and flow temperature, thermal boundary layer is developed
- Boundary layer phenomenon governs an important role in convective heat transfer mode
- In convection mode, sensible heat and latent heat exchange are feasible
- **Convective heat transfer coefficient depends upon boundary layer**

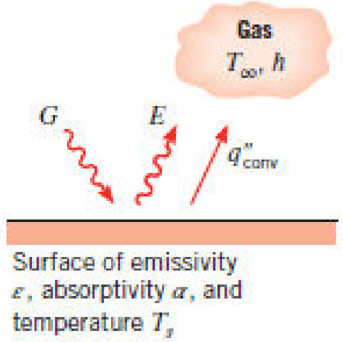


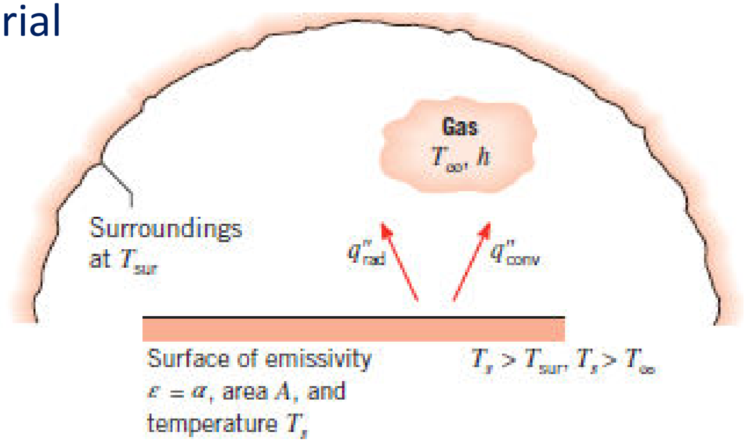


$$q'' = h(T_s - T_\infty)$$

Physical origin of rate equation – thermal radiation

- Regardless of the form of matter, thermal energy emitted at non-zero temperature is attributed to change in electronic configuration of constituent atoms or molecules
- Radiation emitted by the surface originates from thermal energy of the matter bounded by the surface and the rate at which energy is released is termed as surface emissive power . As per Stefan Boltzmann's law for a black body and real surfaces are respectively: $E_b = \sigma T_s^4$ $E = \varepsilon \sigma T_s^4$
- Radiation may also be incident on a surface from its surroundings and the rate at which radiation is incident is termed as irradiation, G $G_{\text{abs}} = \alpha G$
- α depends upon nature of irradiation and surface itself. While ε depends on surface and finish of the material





Physical origin of rate equation – thermal radiation

- A special case that occurs frequently in engineering involves radiation exchange between small surface at T_s and much larger surface that completely surrounds small one
- For such condition, irradiation can be approximated as emission from blackbody at T_{surr}, $G = \sigma T^4_{Surr}$
- If the surface is assumed to have $\alpha = \varepsilon$ then net radiation heat transfer from the surface expressed per unit area of the surface is:

$$q''_{\text{rad}} = \frac{q}{A} = \varepsilon E_b(T_s) - \alpha G = \varepsilon\sigma(T_s^4 - T_{\text{sur}}^4)$$

- For many applications it is convenient to express net radiation exchange in the form: $q_{\text{rad}} = h_r A(T_s - T_{\text{sur}})$
- The radiation heat transfer coefficient h_f is :

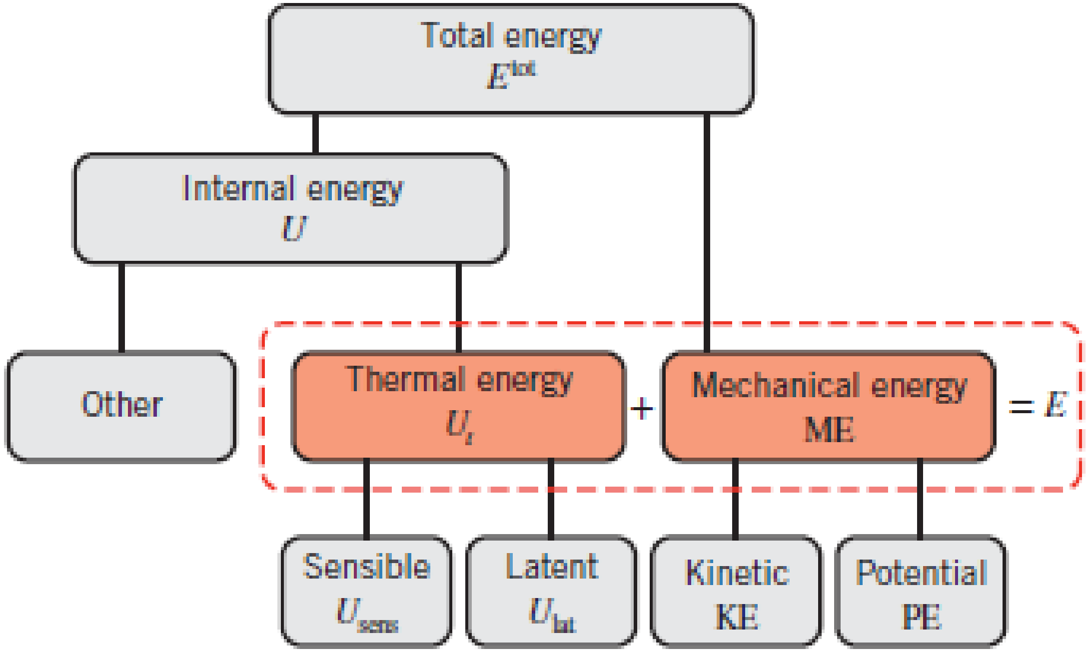
$$h_r \equiv \varepsilon\sigma(T_s + T_{\text{sur}})(T_s^2 + T_{\text{sur}}^2)$$

Relationship to first law of thermodynamics

- For a closed system , first law of thermodynamics states that

$$\Delta E_{st}^{tot} = Q - W$$

where ΔE^{tot} is the change in the total energy stored in the system, Q is the *net* heat transferred to the system, and W is the *net* work done by the system



Relationship to first law of thermodynamics

1. Conservation of Total Energy: First Law of Thermodynamics over a Time Interval (Δt)

The increase in the amount of energy stored in a control volume must equal the amount of energy that enters the control volume, minus the amount of energy that leaves the control volume.

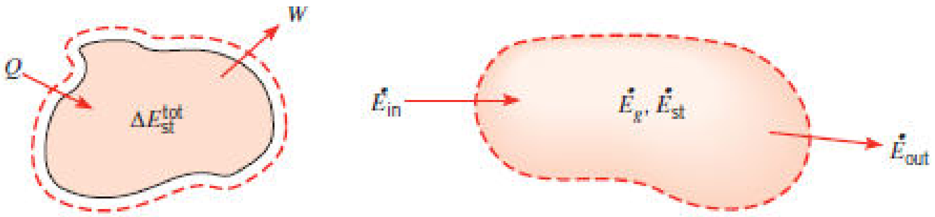
Sum of thermal and mechanical energy is not conserved

2. Conservation of Thermal and Mechanical Energy over a Time Interval (Δt)

The increase in the amount of thermal and mechanical energy stored in a control volume must equal the amount of thermal and mechanical energy that enters the control volume, minus the amount of thermal and mechanical energy that leaves the control volume, plus the amount of thermal and mechanical energy that is generated within the control volume.

3. Conservation of Thermal and Mechanical Energy at an Instant (t)

The rate of increase of thermal and mechanical energy stored in a control volume must equal the rate at which thermal and mechanical energy enters the control volume, minus the rate at which thermal and mechanical energy leaves the control volume, plus the rate at which thermal and mechanical energy is generated within the control volume.

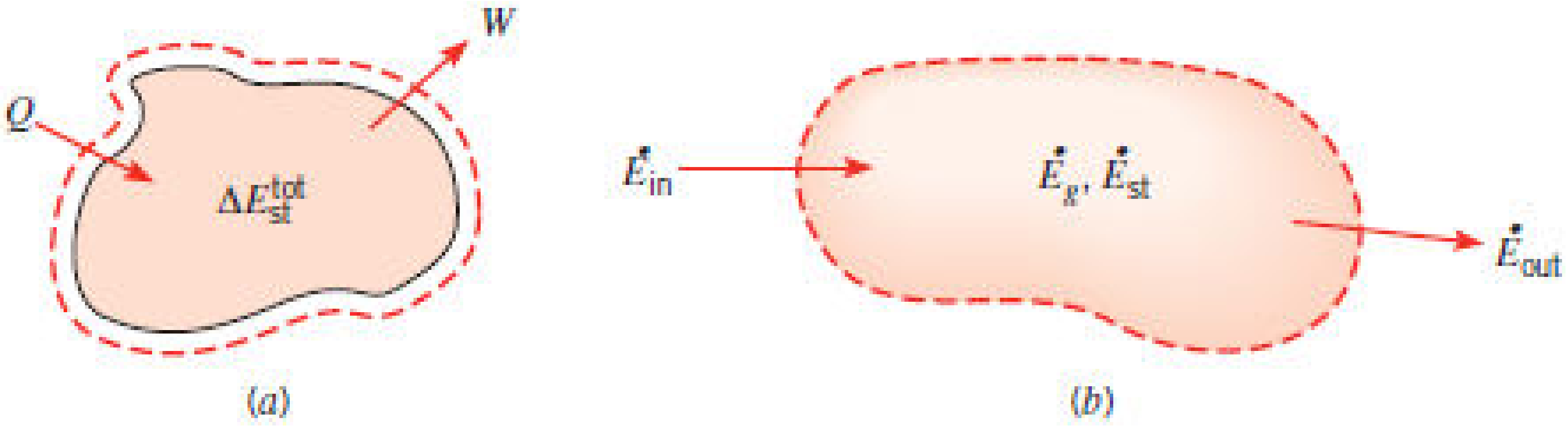


(a) (b)

Relationship to first law of thermodynamics

$$\Delta E_{st} = E_{in} - E_{out} + E_g$$

$$\dot{E}_{st} \equiv \frac{dE_{st}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g$$



First law of thermodynamics application to control volume

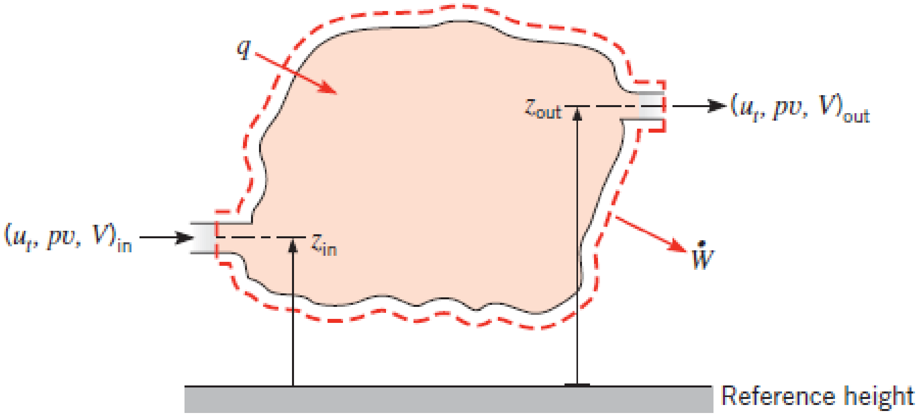
$$\dot{E}_{\mathrm{st}} \equiv \frac{dE_{\mathrm{st}}}{dt} = \dot{E}_{\mathrm{in}} - \dot{E}_{\mathrm{out}} + \dot{E}_{g}$$

- ❖ Under steady state operation with no thermal and mechanical energy generation, the above equation reduces to steady flow energy equation

$$\dot{m}(u_t + pv + \tfrac{1}{2}V^2 + gz)_{\mathrm{in}} - \dot{m}(u_t + pv + \tfrac{1}{2}V^2 + gz)_{\mathrm{out}} + q - \dot{W} = 0$$

- ❖ Since the sum of thermal energy and flow work is enthalpy, and for incompressible fluid steady flow thermal energy equation is given by:

$$q = \dot{m}c_p(T_{\mathrm{out}} - T_{\mathrm{in}})$$



Relationship to second law of thermodynamics and efficiency of heat engine

- Heat engine is any device that operates continuously or cyclically and that converts heat to work
- Examples : internal combustion engines, power plants, and thermoelectric devices
- Second law of thermodynamics states that:
- ***It is impossible for any system to operate in a thermodynamic cycle and deliver a net amount of work to its surroundings while receiving energy by heat transfer from a single thermal reservoir***

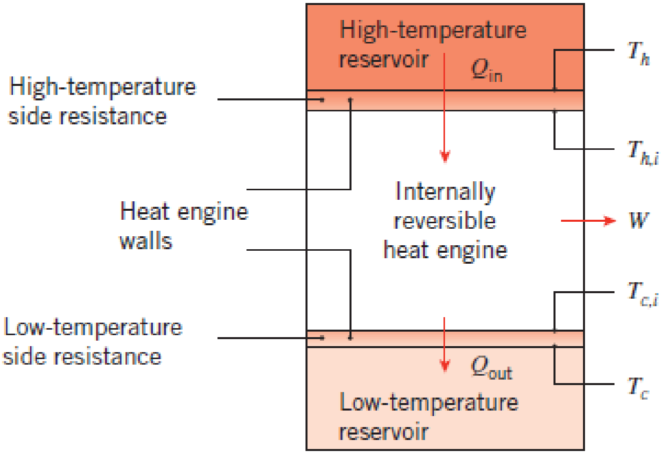
$$\eta \equiv \frac{W}{Q_{\mathrm{in}}} = \frac{Q_{\mathrm{in}} - Q_{\mathrm{out}}}{Q_{\mathrm{in}}} = 1 - \frac{Q_{\mathrm{out}}}{Q_{\mathrm{in}}}$$

$$\eta_C = 1 - \frac{T_c}{T_h}$$

- Any real heat engine, which will necessarily undergo an irreversible process, will have an efficiency lower than η_c
- For heat transfer to occur, there must be a nonzero temperature difference between the reservoir and the heat engine. This reality introduces irreversibility and reduces the efficiency

Relationship to second law of thermodynamics and efficiency of heat engine

- Consider a more realistic model of a heat engine in which heat is transferred into the engine through a thermal resistance $R_{t,h}$ and heat is extracted through second thermal resistance $R_{t,c}$
- Thermal resistances are associated with heat transfer between the heat engine and the reservoirs across a finite temperature difference, by way of the mechanisms of conduction, convection, and/or radiation
- $T_{h,i} < T_h$ and $T_{c,i} > T_c$



- Modified efficiency that accounts for realistic (irreversible) heat transfer processes – η_m

$$\eta_m = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}} = 1 - \frac{q_{\text{out}}}{q_{\text{in}}} = 1 - \frac{T_{c,i}}{T_{h,i}}$$

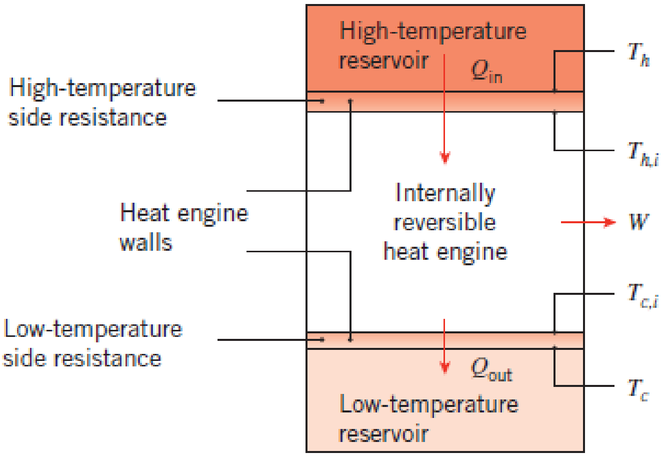
- As the ratio of heat *flows* over a time interval, Q_{out}/Q_{in}, has been replaced by the corresponding ratio of heat *rates*, q_{out}/q_{in}

Relationship to second law of thermodynamics and efficiency of heat engine

- Utilizing thermal heat resistance concept, the heat transfer rates into and out of engine are :

$$q_{\text{in}} = (T_h - T_{h,i})/R_{t,h}$$

$$q_{\text{out}} = (T_{c,i} - T_c)/R_{t,c}$$



- Solving for internal temperatures:

$$T_{h,i} = T_h - q_{\text{in}} R_{t,h}$$

$$T_{c,i} = T_c + q_{\text{out}} R_{t,c} = T_c + q_{\text{in}}(1 - \eta_m) R_{t,c}$$

- More realistic modified efficiency is:

$$\eta_m = 1 - \frac{T_{c,i}}{T_{h,i}} = 1 - \frac{T_c + q_{\text{in}}(1 - \eta_m) R_{t,c}}{T_h - q_{\text{in}} R_{t,h}}$$

$$\eta_m = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}} = 1 - \frac{q_{\text{out}}}{q_{\text{in}}} = 1 - \frac{T_{c,i}}{T_{h,i}}$$

- Solving for η_m:

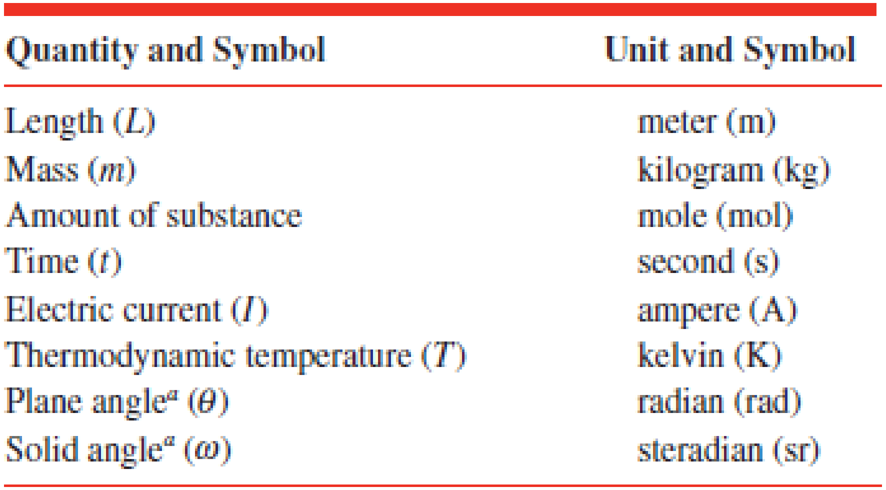
$$\eta_m = 1 - \frac{T_c}{T_h - q_{\text{in}} R_{\text{tot}}}$$

where $R_{tot} = R_{t,h} + R_{t,c}$

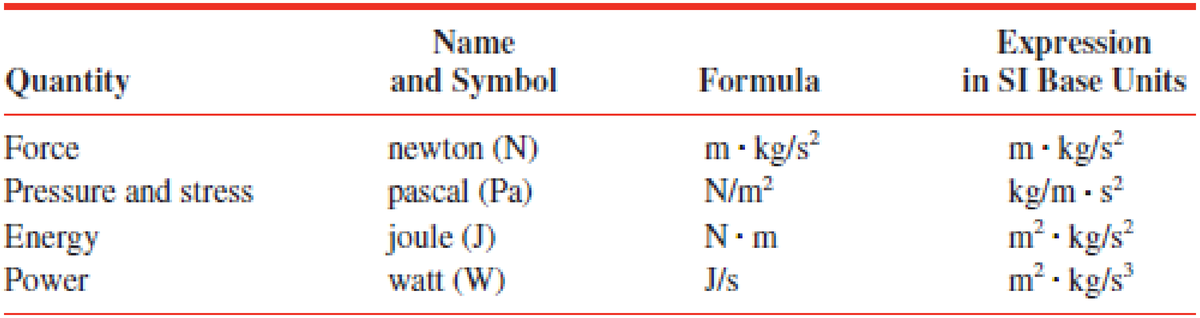
Power output of heat engine is:

$$\dot{W} = q_{\text{in}} \eta_m = q_{\text{in}} \left[1 - \frac{T_c}{T_h - q_{\text{in}} R_{\text{tot}}} \right]$$

A quick note on Units and dimensions



Quantity and Symbol	Unit and Symbol
Length (L)	meter (m)
Mass (m)	kilogram (kg)
Amount of substance	mole (mol)
Time (t)	second (s)
Electric current (I)	ampere (A)
Thermodynamic temperature (T)	kelvin (K)
Plane angle[a] (θ)	radian (rad)
Solid angle[a] (ω)	steradian (sr)



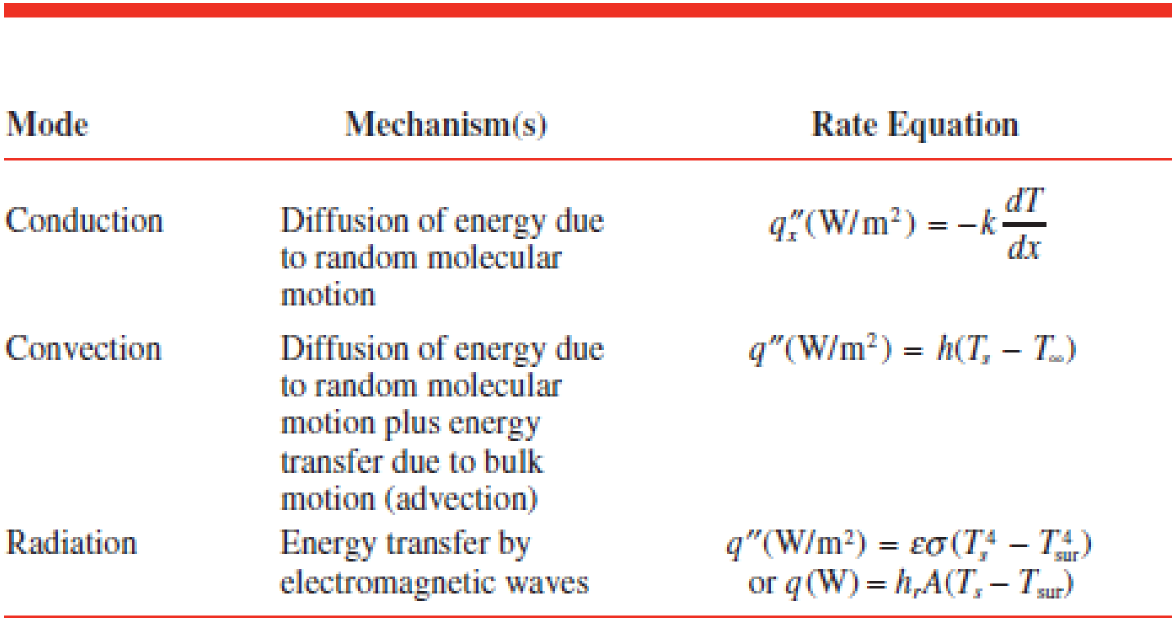
Quantity	Name and Symbol	Formula	Expression in SI Base Units
Force	newton (N)	$\mathrm{m \cdot kg/s^2}$	$\mathrm{m \cdot kg/s^2}$
Pressure and stress	pascal (Pa)	$\mathrm{N/m^2}$	$\mathrm{kg/m \cdot s^2}$
Energy	joule (J)	$\mathrm{N \cdot m}$	$\mathrm{m^2 \cdot kg/s^2}$
Power	watt (W)	$\mathrm{J/s}$	$\mathrm{m^2 \cdot kg/s^3}$

Analysis of Heat transfer problems - Methodology

- ***Known***: After carefully reading the problem, state briefly and concisely what is known about the problem. Do not repeat the problem statement.
- ***Find***: State briefly and concisely what must be found
- ***Schematic***: Draw a schematic of the physical system. If application of the conservation laws is anticipated, represent the required control surface or surfaces by dashed lines on the schematic. Identify relevant heat transfer processes by appropriately labeled arrows on the schematic
- ***Assumptions***: List all pertinent simplifying assumptions
- ***Properties***: Compile property values needed for subsequent calculations and identify the source from which they are obtained
- ***Analysis***: Begin your analysis by applying appropriate conservation laws, and introduce rate equations as needed. Develop the analysis as completely as possible before substituting numerical values. Perform the calculations needed to obtain the desired results
- ***Comments***: Discuss your results. Such a discussion may include a summary of key conclusions, a critique of the original assumptions, and an inference of trends obtained by performing additional *what-if* and *parameter sensitivity* calculations

Applications of heat transfer

- Energy conservation
- Gas turbines
- Cooling of electronic equipment
- Bio-medical engineering



Mode	Mechanism(s)	Rate Equation
Conduction	Diffusion of energy due to random molecular motion	$q''_x(\text{W/m}^2) = -k\frac{dT}{dx}$
Convection	Diffusion of energy due to random molecular motion plus energy transfer due to bulk motion (advection)	$q''(\text{W/m}^2) = h(T_s - T_\infty)$
Radiation	Energy transfer by electromagnetic waves	$q''(\text{W/m}^2) = \varepsilon\sigma(T_s^4 - T_{\text{sur}}^4)$ or $q(\text{W}) = h_r A(T_s - T_{\text{sur}})$

HEAT TRANSFER

MODULE 1
1D STEADY STATE HEAT CONDUCTION

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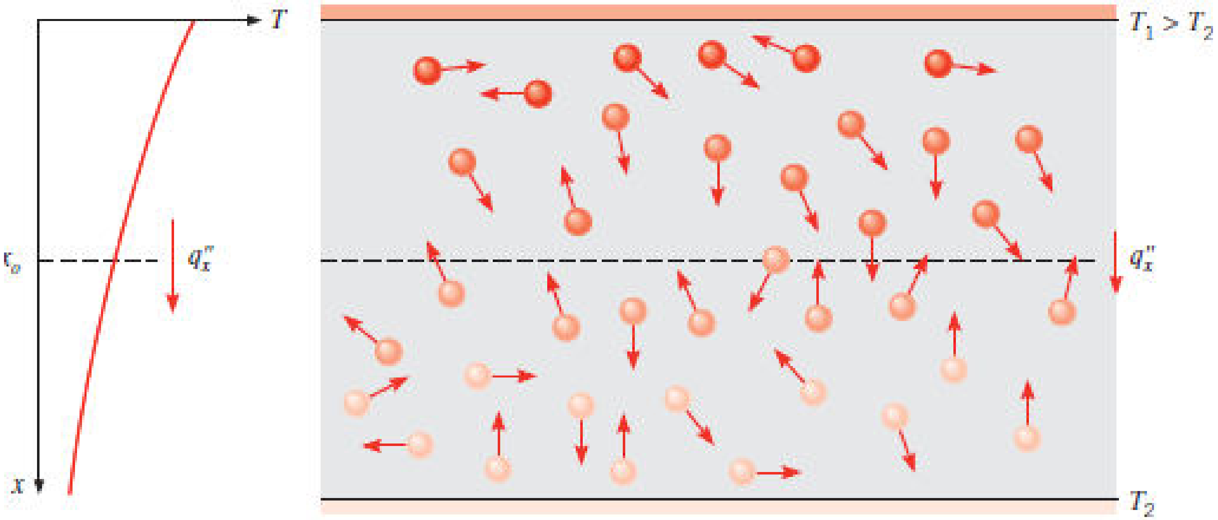
1D Conduction

From the previous lecture it was introduced that:

- Conduction is the transfer of energy from more energetic particle to less energetic particle due to their mutual interaction
- Temperature at a point in medium corresponds to random translation motion, rotation and vibrational motion of molecules
- Energy transfer occurs either by collision of molecules or by random motion in the presence of temperature gradients
- In solids, thermal energy transfer occurs due to lattice vibrations and translational motion of free electrons



1D Conduction

- Heat transfer is governed by Fourier's law and in order to determine heat flux knowledge of temperature distribution is essential
- Fourier's law is applicable to 1D, 3D transient multidimensional conduction in complex geometries

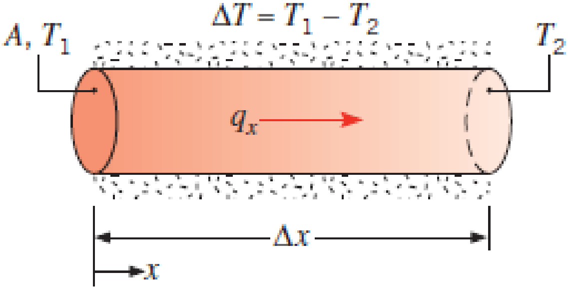
The objectives of this lecture are:

- What is the fundamental origin of Fourier's law?
- What form does it take for different geometries?
- How does thermal conductivity depends upon nature of medium?
- *Our primary objective is to derive heat equation from basic principles which governs temperature distribution in the medium in which conduction is the only mode of heat transfer*
- *Solution to heat equation provides knowledge of temperature distribution that is subsequently used in Fourier law to determine heat flux*

Conduction rate equation

- Fourier's law is phenomenological – is derived from observed phenomena rather than being derived from first principles

A cylindrical rod of known material is insulated on its lateral surface, while its end faces are maintained at different temperatures, with $T_1 > T_2$



We are able to measure the heat transfer rate q_x*, and seek to determine how* q_x *depends on the following variables:* ΔT*, the temperature difference;* Δx*, the rod length; and A, the cross-sectional area*

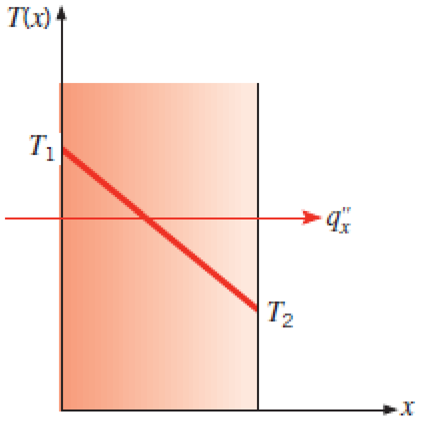
$$q_x \propto A\frac{\Delta T}{\Delta x}$$

On changing the material from metal to plastic Fourier's law is still valid however q_x *is smaller in this case*

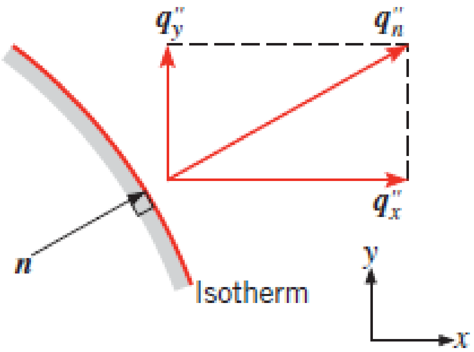
$$q_x = kA\frac{\Delta T}{\Delta x} \rightarrow q_x = -kA\frac{dT}{dx} \rightarrow q''_x = \frac{q_x}{A} = -k\frac{dT}{dx} \quad \text{— } \textbf{\textit{Eq.1}}$$

k, the thermal conductivity (W/m · K), is an important property of the material

Conduction rate equation



- Eq.1 implies that the heat flux is a directional quantity. In particular, the direction of *q’’_x* is *normal* to the cross-sectional area *A*
- The direction of heat flow will always be normal to a surface of constant temperature, called an *isothermal* surface. Isothermal surfaces are planes normal to x-direction
- As heat flux is a vector quantity, a more genera statement of conduction rate equation (Fourier’s law):



$$q'' = -k\nabla T = -k\left(i\frac{\partial T}{\partial x} + j\frac{\partial T}{\partial y} + k\frac{\partial T}{\partial z} \right)$$

— ***Eq.2***

∇ is the three-dimensional del operator, ***i***, ***j***, and ***k*** are the unit vectors in the *x*, *y*, and *z* directions, and *T*(*x*, *y*, *z*) is the scalar temperature field

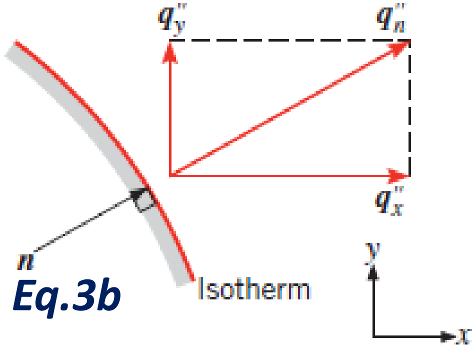
Conduction rate equation

- Heat flux vector is in a direction perpendicular to the isothermal surfaces

$$\mathbf{q}'' = q''_n \mathbf{n} = -k \frac{\partial T}{\partial n} \mathbf{n}$$

— ***Eq.3a***

- q”$_n$ is the heat flux in a direction n, which is normal to an isotherm and when resolved in terms of Cartesian coordinates:



$$\mathbf{q}'' = \mathbf{i}q''_x + \mathbf{j}q''_y + \mathbf{k}q''_z$$

— ***Eq.3b***

$$q''_x = -k\frac{\partial T}{\partial x} \qquad q''_y = -k\frac{\partial t}{\partial y} \qquad q''_z = -k\frac{\partial T}{\partial z}$$

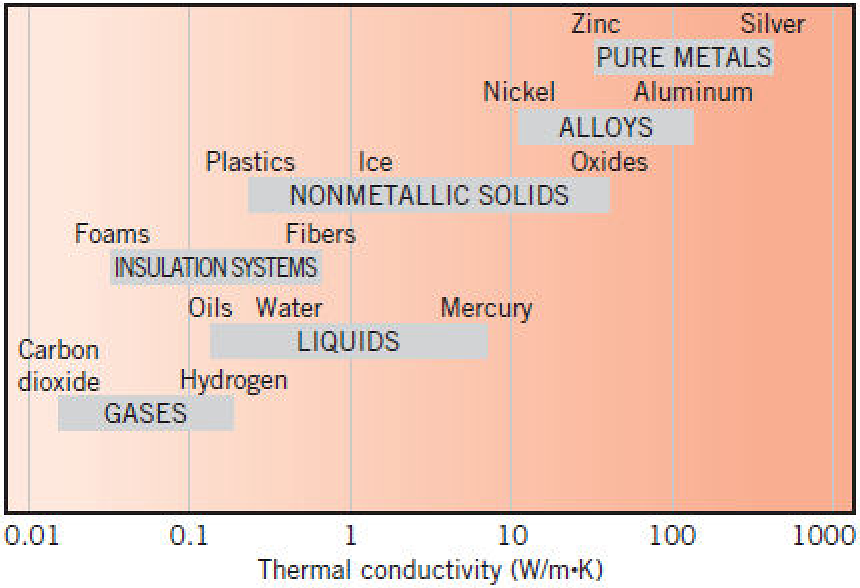
— ***Eq.3c***

- Eq. 2 implies that the medium is isotropic implying that the value of thermal conductivity is independent of coordinate direction. Fourier’s law is valid for all matter regardless of its state (solid, liquid or gas)

Thermal properties of matter

- $k_{solid} >> k_{liquid} >> k_{gas}$
- In solids the transport of thermal energy is due to : migration of free electrons and lattice vibrational waves
- In pure metals, the *electron contribution dominates conduction heat transfer*; while in nonconductors and semiconductors *phonon contribution is dominant*
- *From kinetic theory of gases*

$$k = \frac{1}{3} C \bar{c} \lambda_{mfp}$$



For metals: C– electron specific heat per unit volume, c is the mean electron velocity, and $\lambda_{mfp} \equiv \lambda_e$ is the electron mean free path

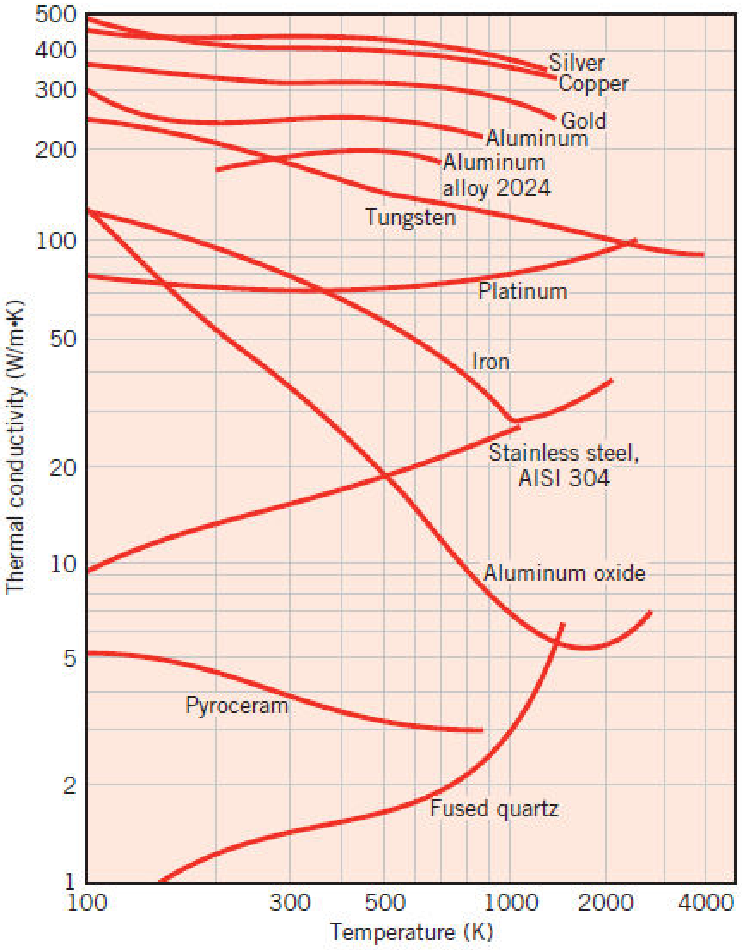
For metals: C– phonon specific heat per unit volume, c is average speed of sound, and $\lambda_{mfp} \equiv \lambda_{phonon}$ is the electron mean free path

Thermal properties of matter

- When electrons and phonons carry thermal energy leading to conduction heat transfer in a solid, the thermal conductivity may be expressed as

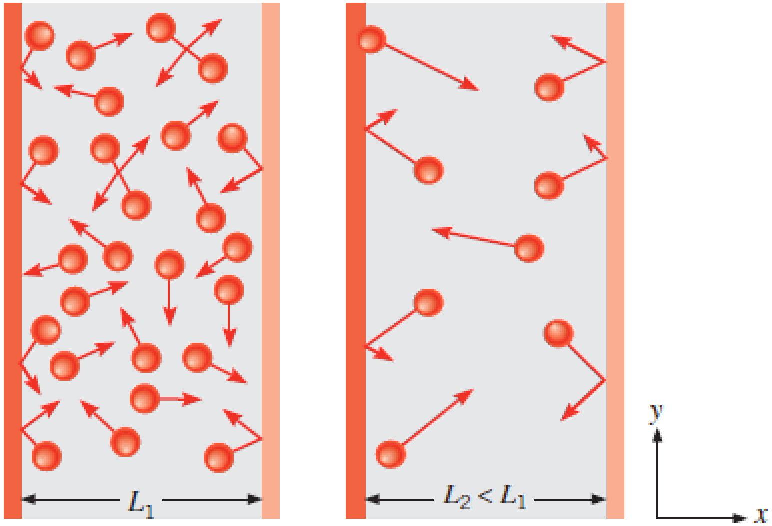
$$k = k_e + k_{\mathrm{ph}}$$

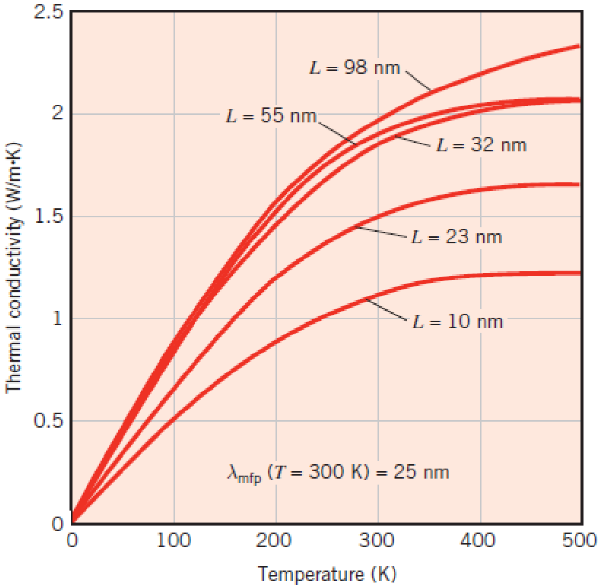
- k_e is inversely proportional to the electrical resistivity, ρ_e. For pure metals, which are of low ρ_e, k_e is much larger than k_{ph}
- For nonmetallic solids, k is determined primarily by k_{ph}, which increases as the frequency of interactions between the atoms and the lattice decreases



Conductivity: Solid state micro and nanoscale effects

- In microelectronics industry, the material's characteristic dimensions can be on the order of micrometers or nanometers and hence significant modification of k occurs
- Cross sections of *films* of the same material having thicknesses L_1 and L_2 are shown in Figure
- L/λ_{mfp} has significant effect on k and hence k decreases as the above ratio decreases



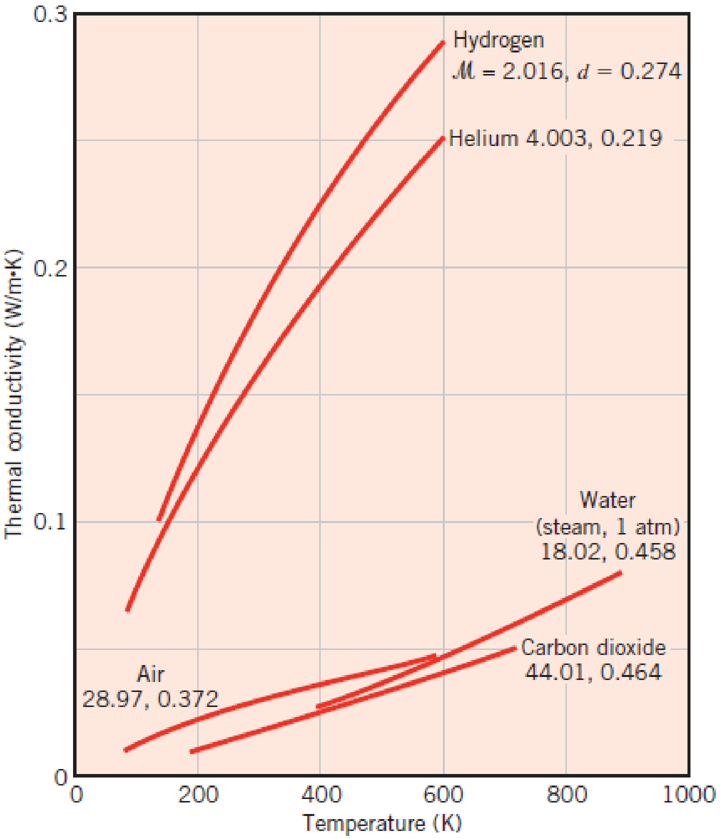


Thermal conductivity in fluid state

- In both liquids and gases the intermolecular spacing is more random than solid: thermal energy transport is less effective
- From kinetic theory of gases , k is directly proportional to density of gas ρ, mean molecular speed c, and mean free path, λ_{mfp}, is the average distance travelled by a molecule before collision is:

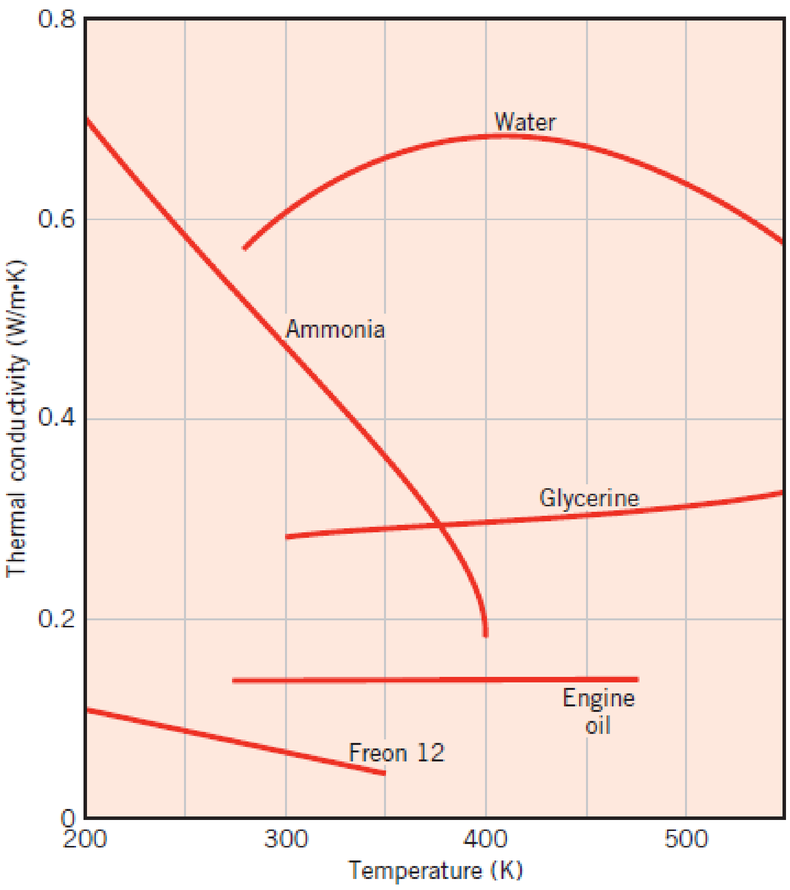
$$k \approx \frac{1}{3} c_v \rho \bar{c} \lambda_{\text{mfp}}$$

- It is presumed that k is independent of gas pressure except when conditions approach perfect vacuum



Thermal conductivity in fluid state

- Molecular conditions associated with liquid are difficult to describe and hence physical mechanisms of k are not well understood
- k of liquids is insensitive to pressure except near thermodynamic critical point
- k generally decreases with increasing molecular weight
- Liquid metals are commonly used in high heat flux applications such as nuclear power plants since their k is much larger than nonmetallic liquids
- In liquids too bulk thermal conductivity is affected when a fluid is constrained by a small physical dimension (L/λ_{mfp})
- Nanofluids are used in applications to tailor k



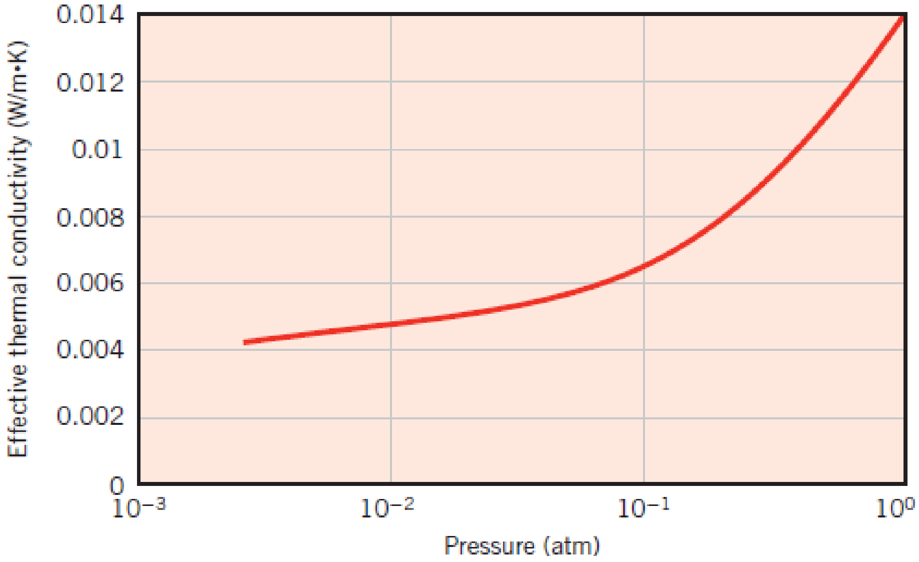
Insulation systems

Thermal insulations consist of low thermal conductivity materials

In conventional *fiber*-, *powder*-, and *flake*-type insulations, the solid material is finely dispersed throughout an air space

A special parameter of the system is its bulk density (solid mass/total volume), which depends strongly on the manner in which the material is packed

Micro- and nanoscale effects can influence the effective k of insulating materials as shown below for nanostructured silica aerogel



Thermophysical properties relevant to heat transfer problems

Thermophysical properties include two distinct categories, transport and thermodynamic properties:

The transport properties are the diffusion rate coefficients such as *k*, the thermal conductivity (for heat transfer), and ν , the kinematic viscosity (for momentum transfer)

Density (ρ) and specific heat (c_p) are two such properties used in thermodynamic analysis

The product ρc_p (J/m^3·K), commonly termed the *volumetric heat capacity*, measures the ability of a material to store thermal energy

Substances of large density are typically characterized by small specific heats, many solids and liquids, which are very good energy storage media

$$\alpha = \frac{k}{\rho c_p}$$

Materials of large will respond quickly to changes in their thermal environment, whereas materials of small will respond more sluggishly, taking longer to reach a new equilibrium condition

HEAT TRANSFER

MODULE 1

1D STEADY STATE HEAT CONDUCTION

Dr. Satish Idury
Associate Professor

Heat Diffusion Equation- significance

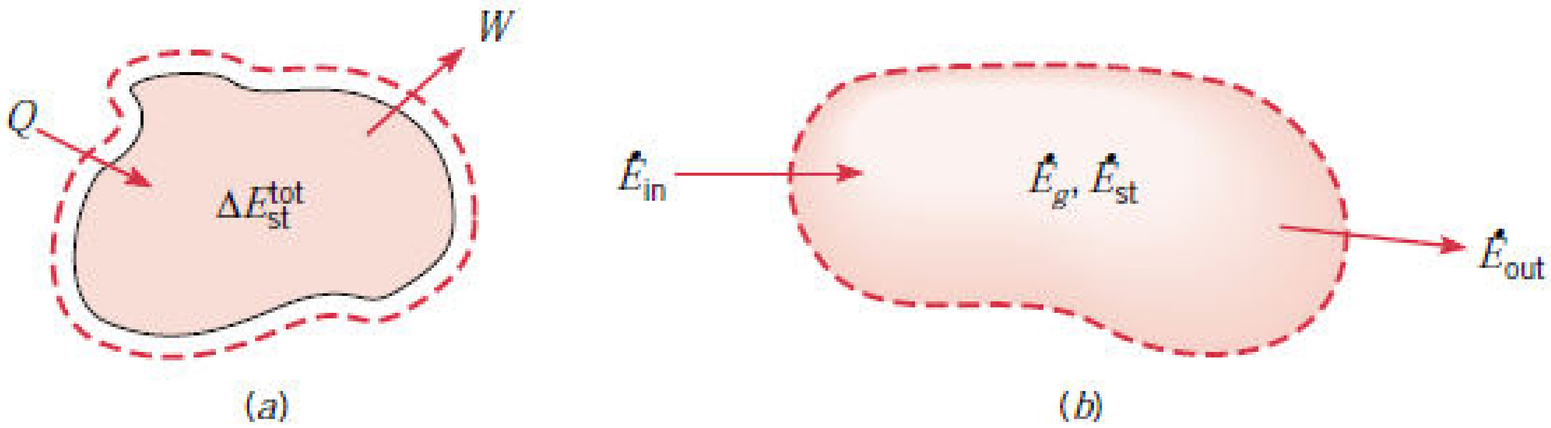
- A major objective in a conduction analysis is to determine the temperature field in a medium resulting from conditions imposed on its boundaries
- Once the temperature distribution is known the conduction heat flux at any point in the medium is computed from Fourier’s law
- Knowledge of temperature distribution can be used to ascertain structural integrity by determining thermal stresses, expansions and deflections
- Temperature distribution knowledge aids to optimize thickness of insulating material or determine compatibility of special coatings or adhesives used with material

Heat Diffusion Equation- methodology

- Apply the energy conservation for a particular control volume based on first law of thermodynamics

- Define a differential control volume, identify the relevant energy transfer processes and introduce appropriate rate equations

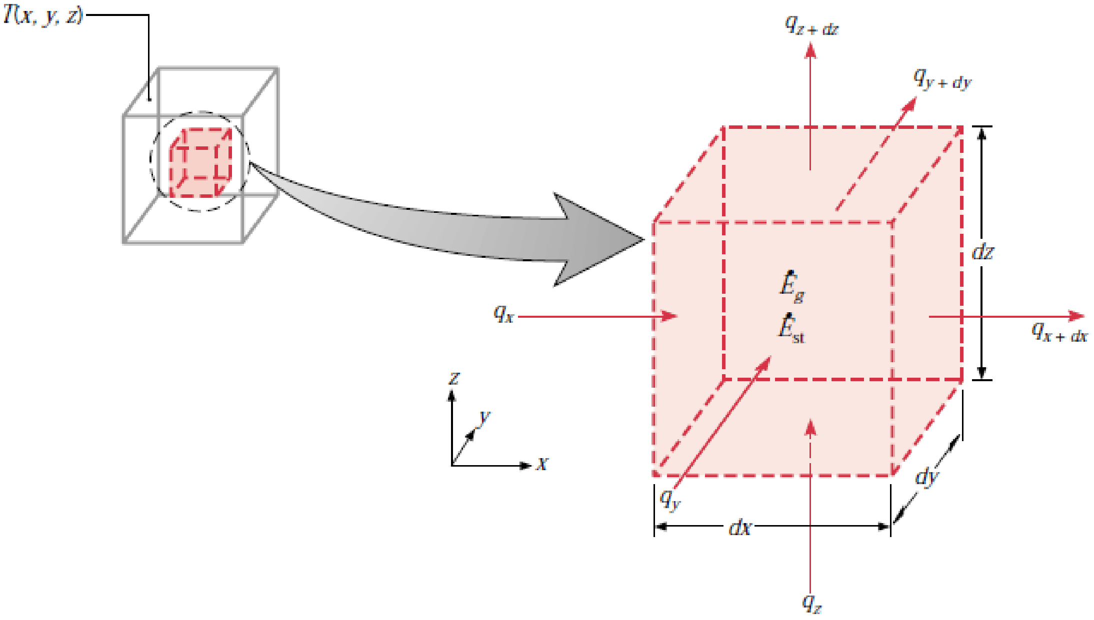
- The resultant derived differential equation whose solution for a prescribed boundary conditions, provides temperature distribution in the medium



Conservation of energy for a) closed system over a time interval b) for control volume at an instant

Heat Diffusion Equation- derivation

- Consider a homogeneous medium within which there is no bulk motion and the temperature distribution T(x, y, z) is expressed in **Cartesian coordinates**
- By incorporating first law at an instant of time, identify the energy process that are relevant to control volume. Incompressibility of the medium is assumed in derivation. There is no mechanical work and only thermal energy is considered
- In case of temperature gradients, conduction heat transfer occurs at control surfaces



Differential control volume , dx, dy, dz in Cartesian coordinates

Heat Diffusion Equation- derivation

➢Conduction heat rates perpendicular to each of the control surfaces at x, y, z coordinate locations are indicated by q_x, q_y and q_z respectively

➢Conduction heat rates at the opposite surfaces can be expressed as Taylor series expansion by neglecting higher order terms

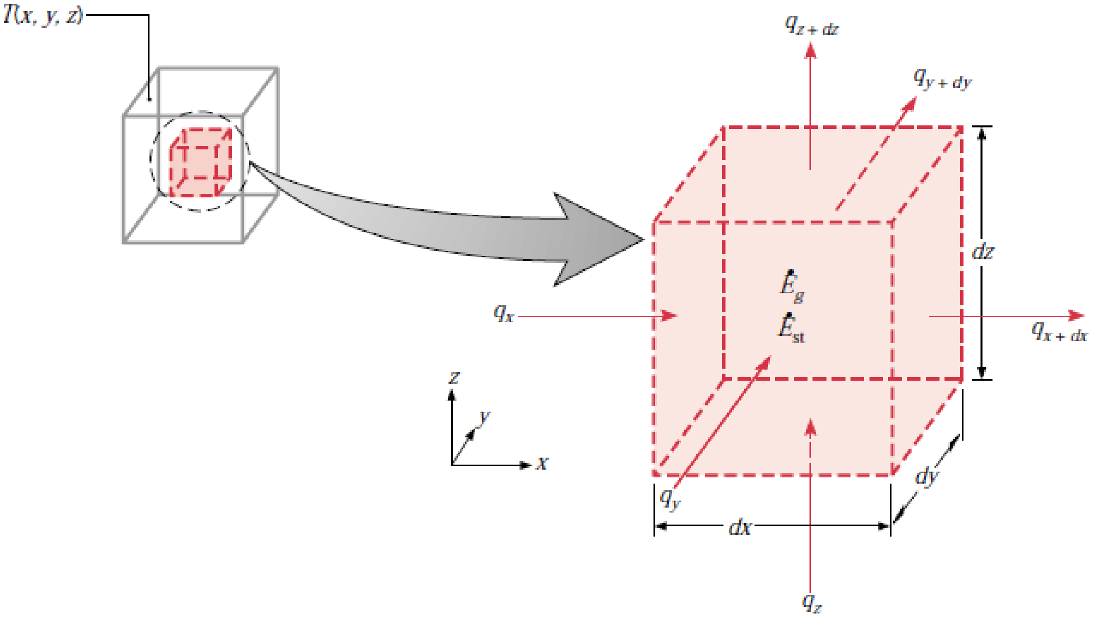
$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x}dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y}dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z}dz$$

Energy generation term , $\dot{E}_g$

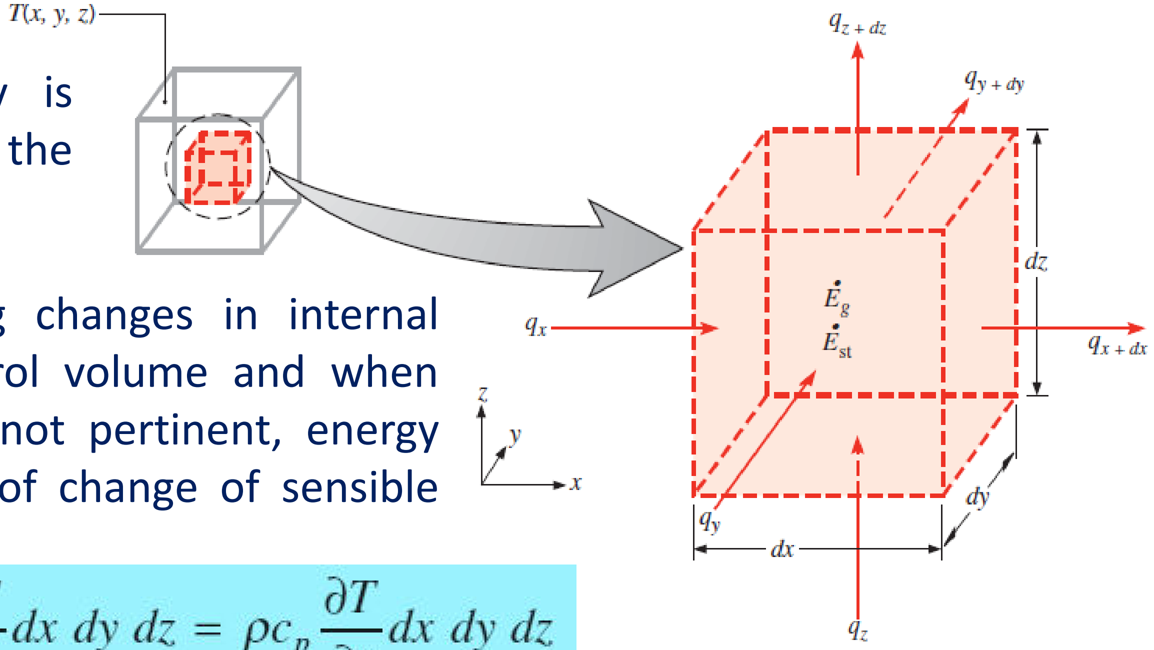
$$\dot{E}_g = \dot{q}\ dx\ dy\ dz$$



Differential control volume , dx, dy, dz in Cartesian coordinates

Heat Diffusion Equation- derivation

$\dot{q}$, the rate at which energy is generated per unit volume of the medium (W/m^3)



If the material is experiencing changes in internal thermal energy stored in control volume and when latent heat energy effects are not pertinent, energy storage term reduces to rate of change of sensible energy:

$$\dot{E}_{st} = \frac{\partial U_{sens}}{\partial t} = \rho c_v \frac{\partial T}{\partial t} dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

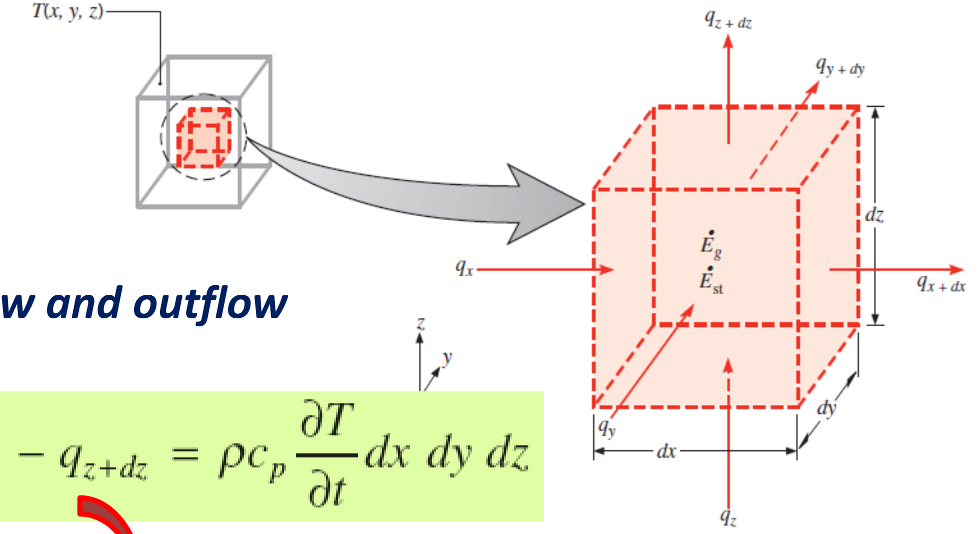
$\dot{E}_g$ and $\dot{E}_{st}$ represent different physical processes

is a manifestation of some energy conversion process involving thermal energy on one hand and some other form of energy, such as chemical, electrical, or nuclear, on the other. It can be positive or negative

Heat Diffusion Equation- derivation

On a rate basis, the general form of the conservation of energy requirement is:

$$\dot{E}_{\text{in}} + \dot{E}_g - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$



Conduction rates constitute energy inflow and outflow

$$q_x + q_y + q_z + \dot{q}\,dx\,dy\,dz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho c_p \frac{\partial T}{\partial t} dx\,dy\,dz$$

Since------

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

Therefore rate equation becomes-----

$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + \dot{q}\,dx\,dy\,dz = \rho c_p \frac{\partial T}{\partial t} dx\,dy\,dz$$

Heat Diffusion Equation- derivation

The conduction heat rates in an isotropic material may be evaluated from Fourier's law

$$q_x = -k\,dy\,dz\frac{\partial T}{\partial x}$$

$$q_y = -k\,dx\,dz\frac{\partial T}{\partial y}$$

$$q_z = -k\,dx\,dy\frac{\partial T}{\partial z}$$

$$-\frac{\partial q_x}{\partial x}dx - \frac{\partial q_y}{\partial y}dy - \frac{\partial q_z}{\partial z}dz + \dot{q}\,dx\,dy\,dz = \rho c_p\frac{\partial T}{\partial t}dx\,dy\,dz$$

After substitution for q_x, q_y and q_z and dividing by dx dy dz through out the equation:

$$\boxed{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}}$$

The above equation is the general form of heat diffusion equation

Heat Diffusion equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

- *Heat diffusion equation or heat equation is the basic tool for conduction analysis as we can obtain temperature distribution T (x, y, z) as function of time*

- *Heat equation describes an important condition of conservation of energy*

$\partial(k\,\partial T/\partial x)/\partial x$ ➡ *is related to net heat conduction flux into the control volume in x-direction*

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx = q''_x - q''_{x+dx}$$

At any point in the medium the net rate of energy transfer by conduction into a unit volume plus the volumetric rate of thermal energy generation must be equal to rate of change of thermal energy stored within the volume

Various forms of heat equation

Basic heat equation: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

For constant thermal conductivity: $$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$\alpha = k/\rho c_p$ *is termed as thermal diffusivity*

Under steady state condition, there is no change in energy storage: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0$$

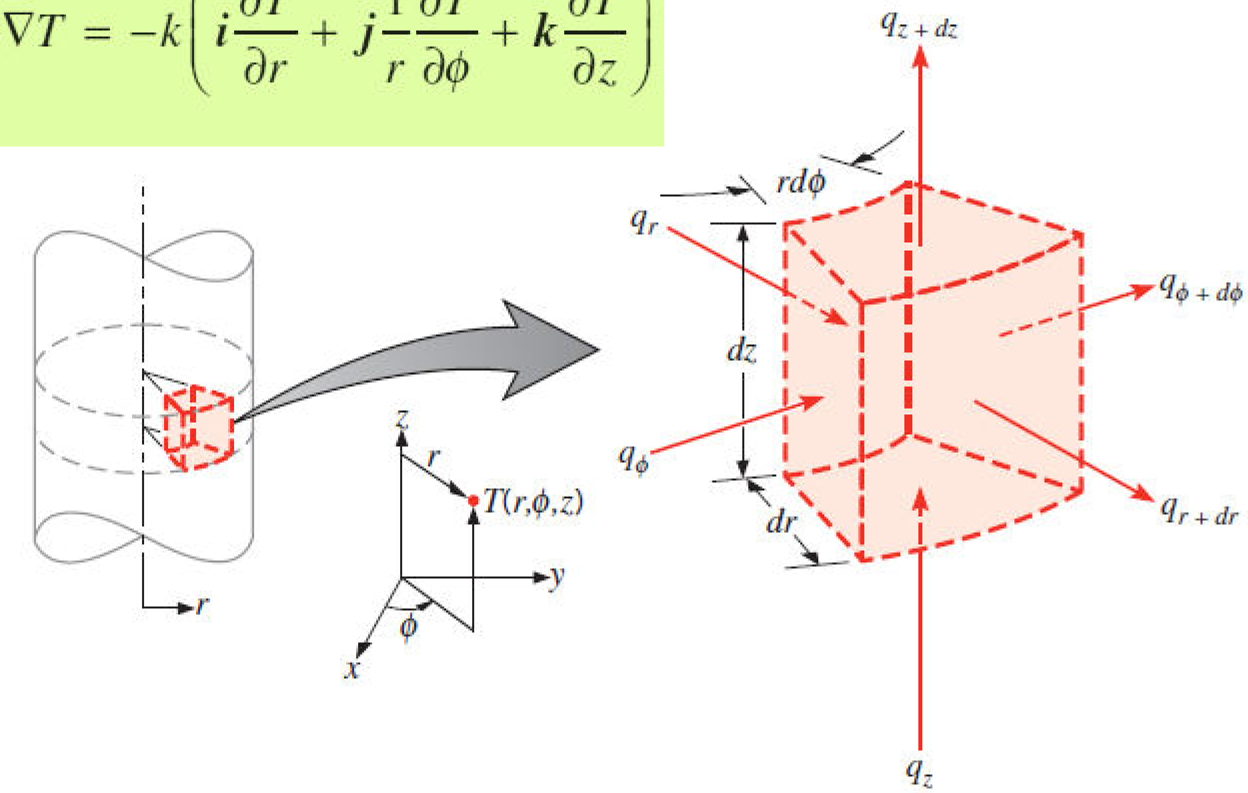
If heat transfer is one dimensional and there is no energy generation : $$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

Under steady state, one dimensional conditions with no energy generation, heat flux is constant in the direction of transfer -------------- $dq_x''/dx = 0$

Heat equation in cylindrical coordinates

When the del operator ∇ *is expressed in cylindrical coordinates, with i, j, k representing unit vectors in r, φ, z directions, the general form Fourier's law is*:

$$q'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \phi} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$



Differential control volume dr. rdϕ. dz for conduction analysis in cylindrical coordinates

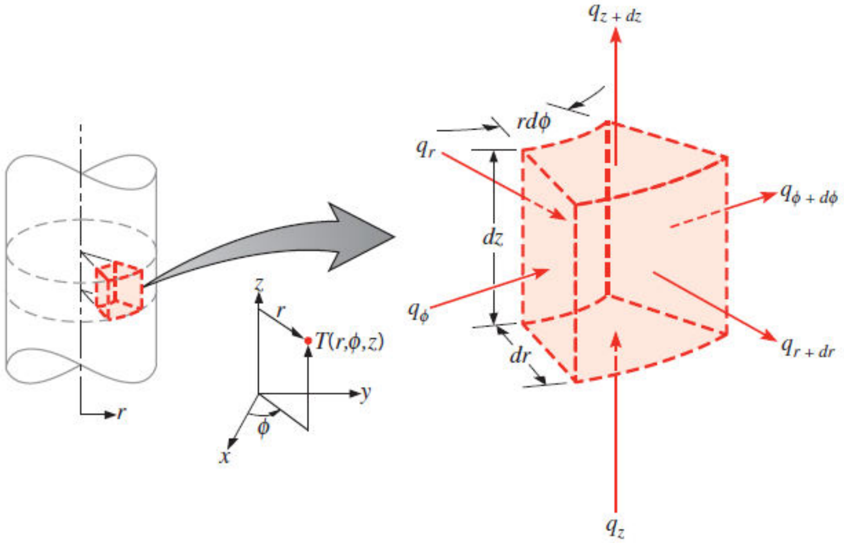
Heat equation in cylindrical coordinates

$$q''_r = -k\frac{\partial T}{\partial r} \qquad q''_\phi = -\frac{k}{r}\frac{\partial T}{\partial \phi} \qquad q''_z = -k\frac{\partial T}{\partial z}$$

represent heat fluxes in radial, circumferential and axial directions respectively

On application of energy balance to differential control volume, the following general form of heat equation is obtained for cylindrical coordinates

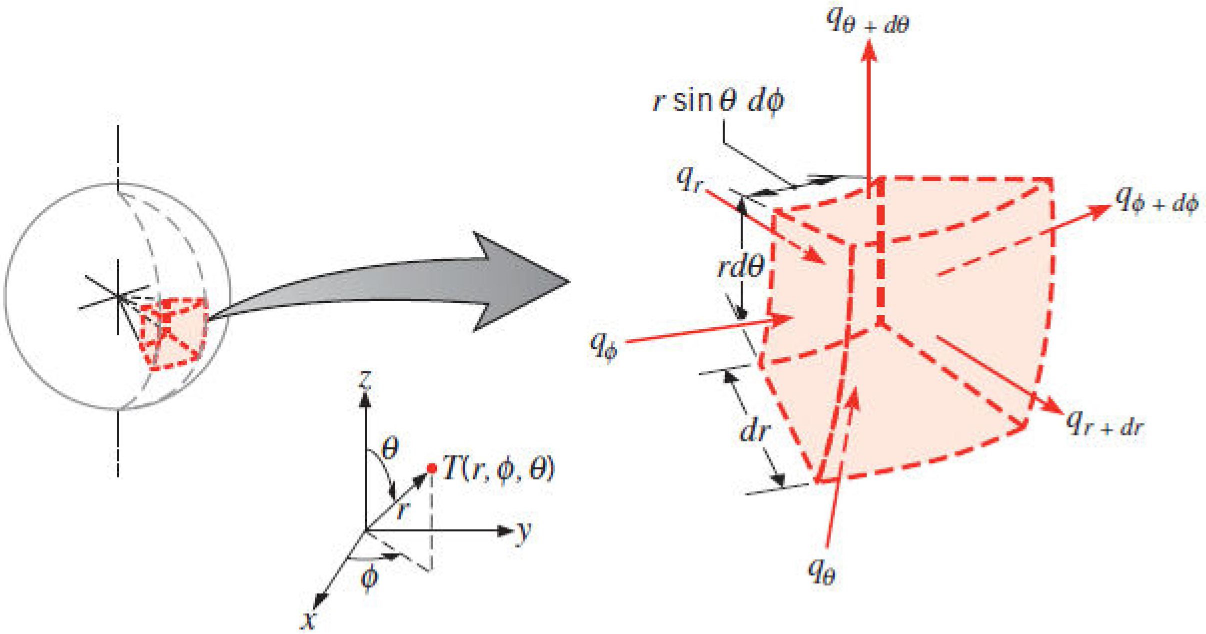
$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$



Heat equation in spherical coordinates

In spherical coordinates with i, j, k representing the unit vectors in r, ϑ, φ directions, the general form of Fourier's law :

$$\boldsymbol{q}'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \theta} + \boldsymbol{k}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$



Differential control volume dr. rsinθ dϕ. rdθ for conduction analysis in spherical coordinates

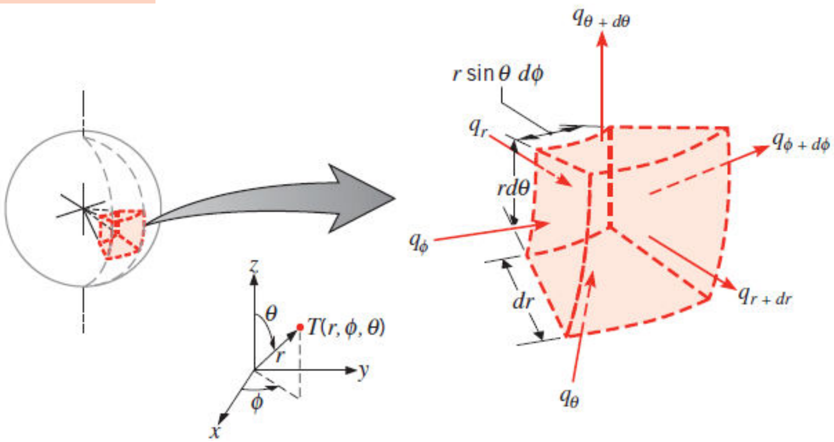
Heat equation in spherical coordinates

$$q_r'' = -k\frac{\partial T}{\partial r} \qquad q_\theta'' = -\frac{k}{r}\frac{\partial T}{\partial \theta} \qquad q_\phi'' = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

heat flux components in the radial, polar, and azimuthal directions, respectively

On application of energy balance to differential control volume, the following general form of heat equation is obtained for spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$



Boundary and initial conditions

- To determine the temperature distribution in a medium, it is necessary to solve the appropriate form of the heat equation

- A solution depends on the physical conditions existing at the *boundaries* of the medium and, if the situation is time dependent, on conditions existing in the medium at some *initial time*

- As heat equation is second order in the spatial coordinates, two boundary conditions must be expressed for each coordinate needed to describe the system

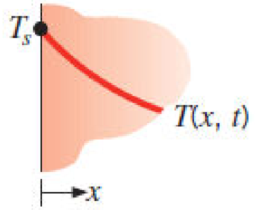
- Since heat equation is first order in time, only one condition, termed the *initial condition*, must be specified

Boundary conditions for heat diffusion equation at surface x = 0 for 1D system

Surface maintained at fixed temperature T_s also termed as Dirichlet condition. Closely approximated when surface in contact with melting solid or boiling liquid

1. Constant surface temperature

$$T(0,t) = T_s$$

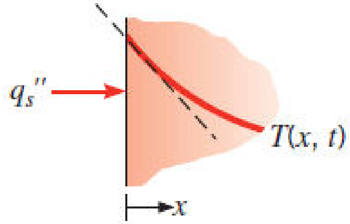


Neumann or second condition corresponds to existence of constant heat flux at surface q_s''. This heat flux is related to Fourier's law at surface. Realized by bonding thin film electric heater to surface

2. Constant surface heat flux

(a) Finite heat flux

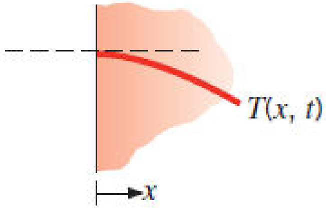
$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_s''$$



A special case of second condition exits for perfectly insulated or adiabatic surface

(b) Adiabatic or insulated surface

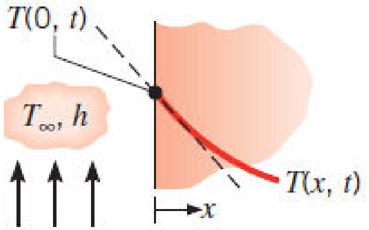
$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$



Boundary condition of third kind corresponds to existence of convective heating or cooling of surface and is obtained from surface energy balance

3. Convection surface condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0,t)]$$



HEAT TRANSFER

MODULE 1
1D STEADY STATE HEAT CONDUCTION

Dr. Satish Idury
Associate Professor

1D steady state conduction – Temp gradient in single coordinate system and temp at each point is independent of time

Our objectives are to determine for 1D steady state conduction case:

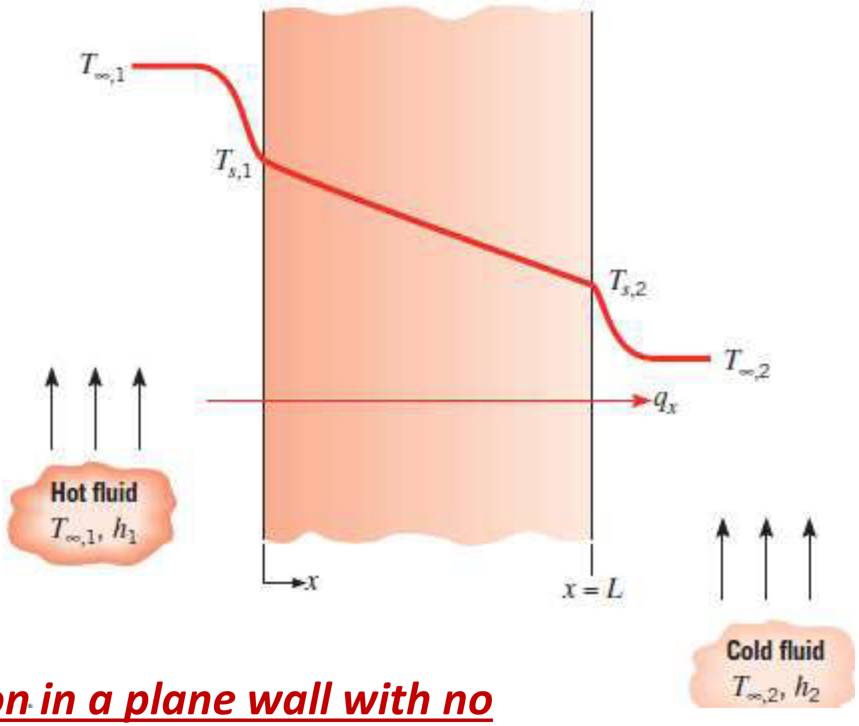
- Heat transfer with no generation of internal energy
- Determine expressions for temperature distribution and heat transfer rate in planar, cylindrical and spherical geometries
- For planar, spherical and cylindrical geometries, introduce the concept of thermal resistance
- We establish that thermal circuits can model heat flow similar to electrical circuits for current flow

1D steady state conduction – Plane wall case

- For 1D conduction in a plane wall, temperature is function of x-coordinate only and heat is transferred exclusively in this direction

- Temp distribution within the wall can be obtained by solving heat equation with appropriate boundary conditions:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$



for one-dimensional, steady-state conduction in a plane wall with no heat generation, the heat flux is a constant, independent of x

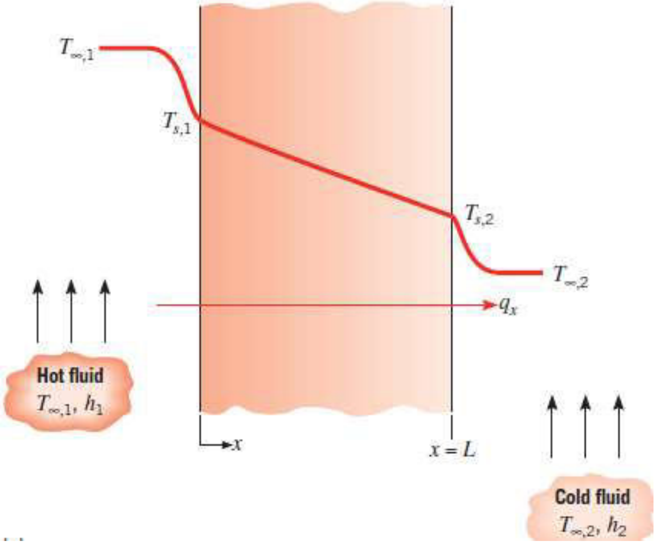
Further if k of wall material is assumed constant , integrating the above equation twice will give general solution:

$$T(x) = C_1x + C_2$$

1D steady state conduction – Plane wall case

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

$$T(x) = C_1 x + C_2$$



To obtain the constants of integration, C_1 and C_2, boundary conditions must be introduced. On applying conditions of the first kind at x = 0 and x = L, in which case

$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

At x = 0, general solution is ***$T_{s,1} = C_2$*** *At x = L,*

$$T_{s,2} = C_1 L + C_2 = C_1 L + T_{s,1}$$

$$\frac{T_{s,2} - T_{s,1}}{L} = C_1$$

Substituting the expression in general solution of temperature distribution

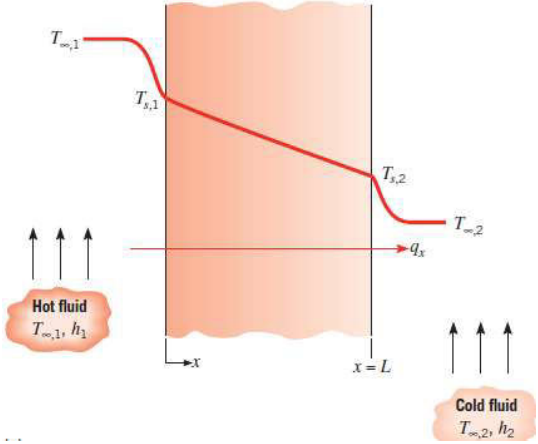
$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1}$$

1D steady state conduction – Plane wall case

- For 1D steady state conduction in plane wall with no heat generation and constant thermal conductivity, ***temperature varies linearly with x***
- To determine heat transfer rate , apply Fourier’s law:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

- For a plane wall heat rate ,q and heat flux, q_x'' are constant and independent of x



Thermal resistance concept

- For 1D steady state conduction in plane wall with no heat generation and constant thermal conductivity, ***temperature varies linearly with x***

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

- *A perfect analogy exists between the diffusion of heat and electrical charge*
- An electrical resistance is associated with the conduction of electricity, a thermal resistance may be associated with the conduction of heat
- Hence we can define resistance as the ratio of a driving potential to the corresponding transfer rate:

$$R_{t,\text{cond}} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}$$

$$R_e = \frac{E_{s,1} - E_{s,2}}{I} = \frac{L}{\sigma A}$$

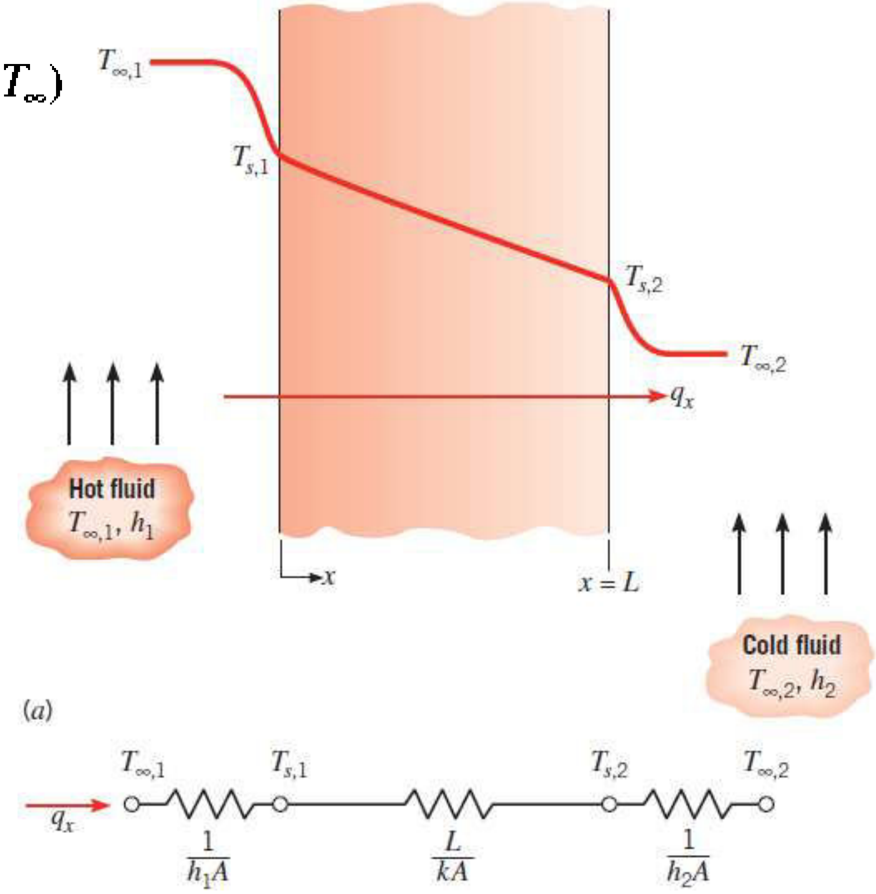
- Recall perfect similarity between Ohm's law for electrical conduction and **heat conduction rate**

Thermal resistance concept

- ❖A thermal resistance is also associated with heat transfer by convection at the surface
- ❖From Newton's law of cooling: $q = hA(T_s - T_\infty)$

- ❖Thermal resistance for convection is then:

$$R_{t,\text{conv}} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$



- ❖ Circuit representations provide a useful tool for both conceptualizing and quantifying heat transfer problems
- ❖ The equivalent thermal circuit for the plane wall with convection surface conditions as shown in figure is:

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

q_x may be determined from separate consideration of each element in the network

Thermal resistance concept

- In terms of overall temperature difference $T_{\infty,1} - T_{\infty,2}$ and total thermal resistance R_{tot}

- Heat transfer rate is expressed as:

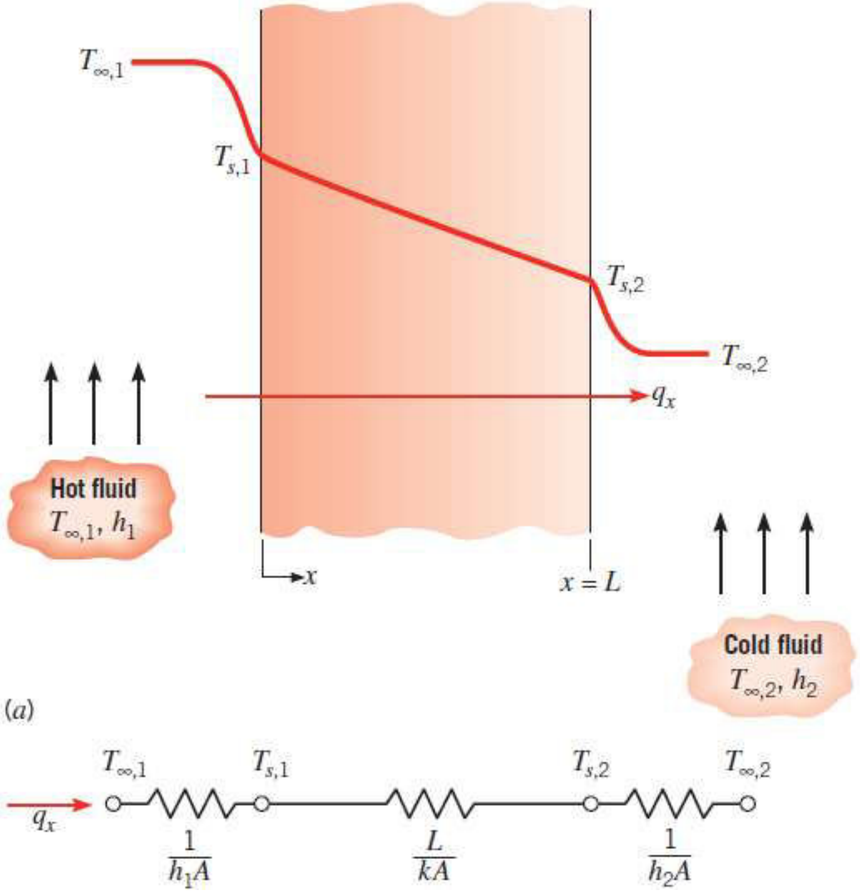
$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$

- As the conduction and convection resistances are in series:

$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

- Radiation exchange between the surface and surroundings is important if convection heat transfer coefficient is small and radiation thermal resistance is:

$$R_{t,rad} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A}$$

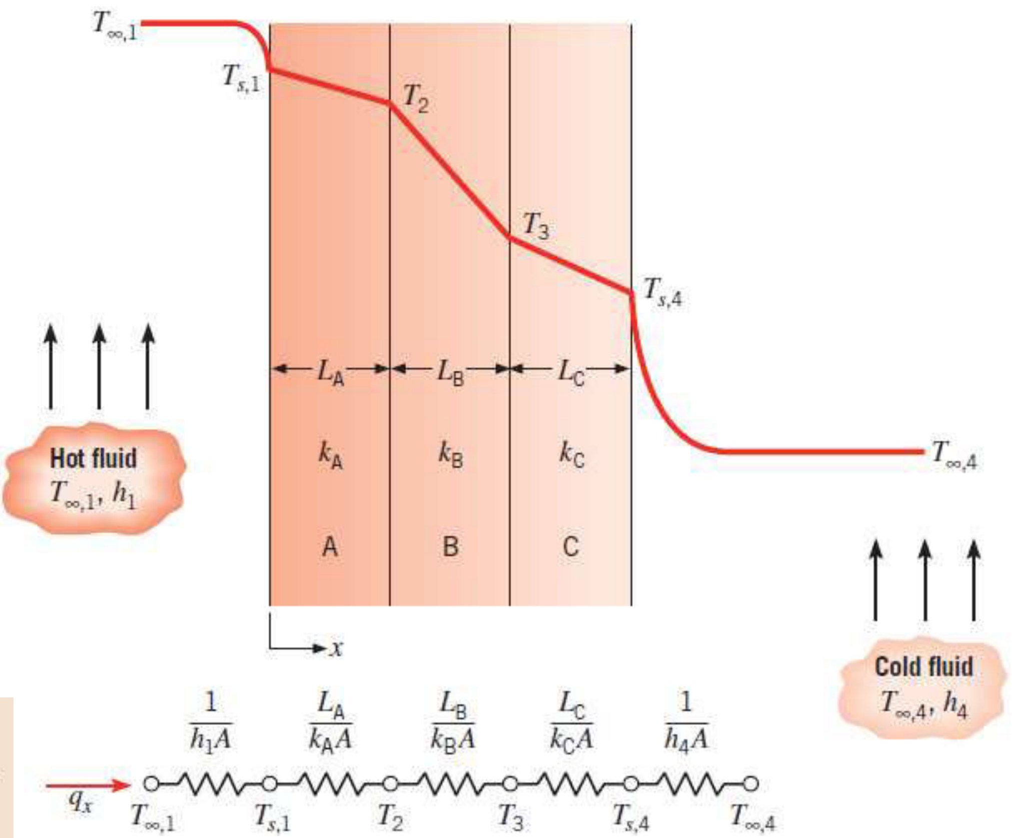


The Composite Wall

- Equivalent thermal circuits for complex systems such as composite walls involves any number of series and parallel resistances due to different layers of thermal material

- 1D heat transfer rate for such system is expressed as:

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\Sigma R_t}$$



$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1 A)} = \frac{T_{s,1} - T_2}{(L_A/k_A A)} = \frac{T_2 - T_3}{(L_B/k_B A)} = \cdots$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1 A) + (L_A/k_A A) + (L_B/k_B A) + (L_C/k_C A) + (1/h_4 A)]}$$

The Composite Wall

❖With composite systems, it is convenient to work with overall heat transfer coefficient is defined by expression analogous to Newton's law of cooling:

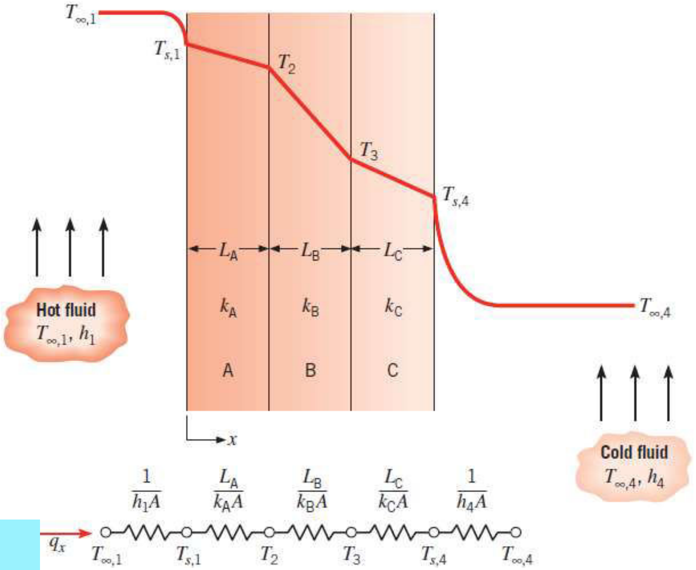
$$q_x \equiv UA\,\Delta T$$

❖Overall heat transfer is related to total thermal resistance :

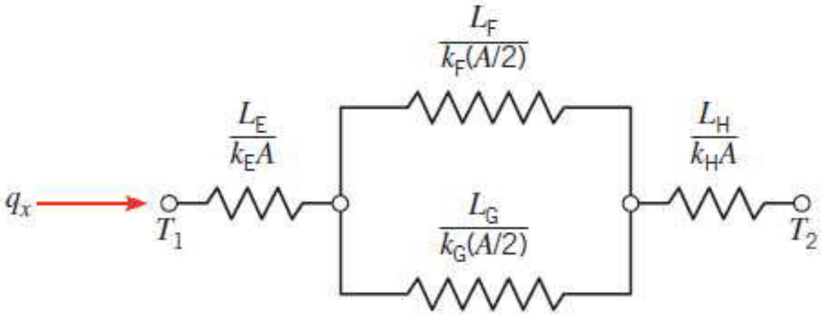
$$UA = \frac{1}{R_{tot}}$$

$$U = \frac{1}{R_{\text{tot}}A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]}$$

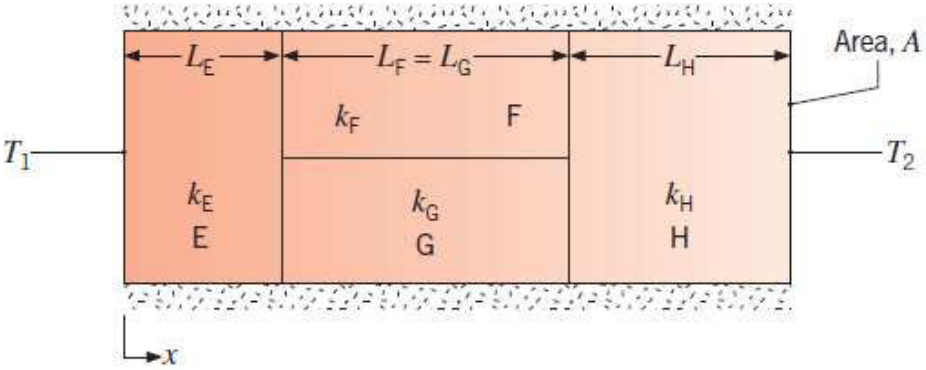
$$R_{\text{tot}} = \Sigma R_t = \frac{\Delta T}{q} = \frac{1}{UA}$$

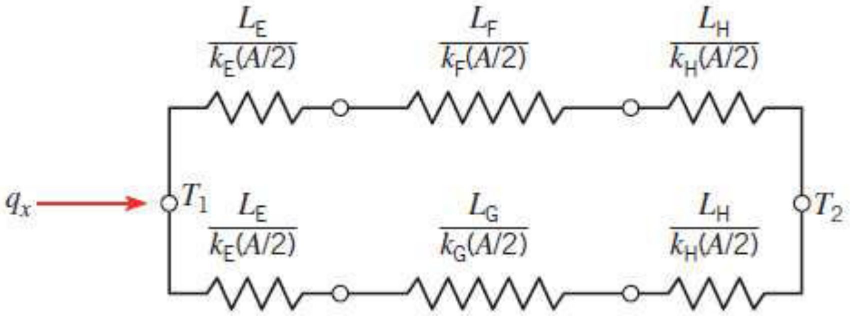


Equivalent thermal circuits for series – parallel composite wall



(a)





(b)

- Although heat flow is multi dimensional it is reasonable to assume 1D conditions
- For case (a) it is presumed that surfaces normal to x direction are isothermal
- For case (b) surfaces parallel to x-direction are adiabatic

Different results are obtained for R_{tot}, and the corresponding values of q bracket the actual heat transfer rate. These differences increase with increasing $|k_F - k_G|$, as multidimensional effects become more significant

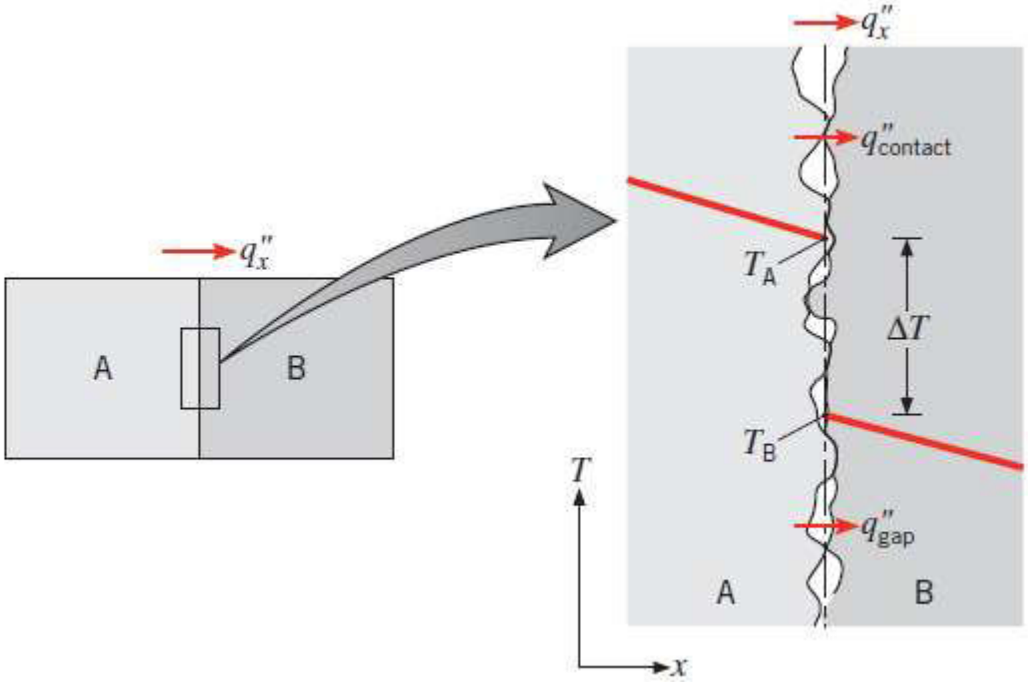
Contact Resistance- Temperature drop

❖ In Composite systems, the temperature drop across the interface between materials may be appreciable

❖Temperature drop is attributable to thermal contact resistance, $R_{t,c}$

❖For a unit area of the interface, the resistance is defined as:

$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$



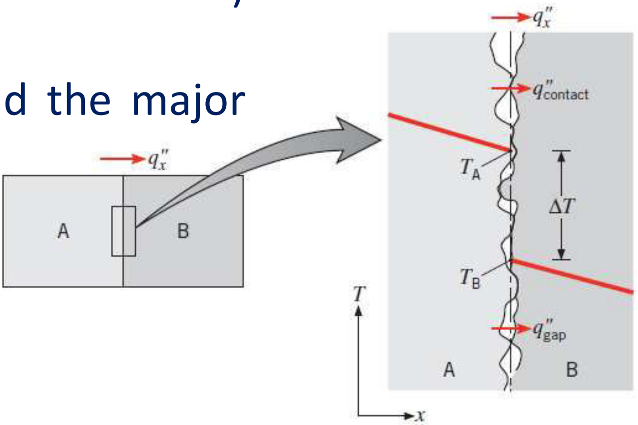
The existence of thermal contact resistance is principally due to surface roughness effects

❖Heat transfer is due to conduction across actual contact area and due to conduction and or radiation across the gaps

Contact Resistance- Temperature drop

- Contact resistance may be viewed as two parallel resistances: a) due to contact spots b) due to gaps
- For rough surfaces, contact area is typically small and the major contribution of resistance is made by gaps

- For solids whose k exceeds that of interfacial fluid, the contact resistance may be reduced by increasing area of contact spots

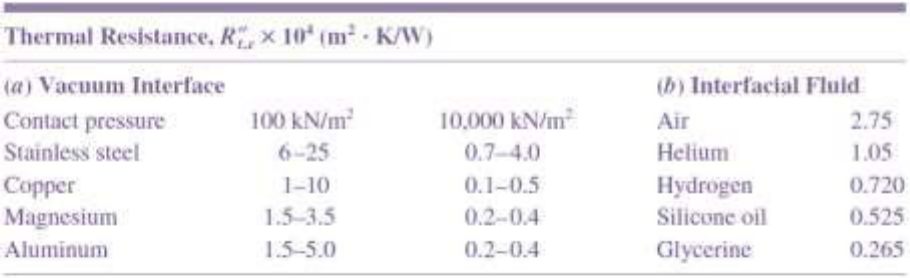


- Contact resistance can also be reduced by increasing contact pressure, reducing the roughness of mating surfaces, selecting interfacial fluid of large k

- If the characteristic gap width L, becomes small, L/λ_{mfp}, becomes small . Hence k of interfacial gap is reduced by micro scale effects

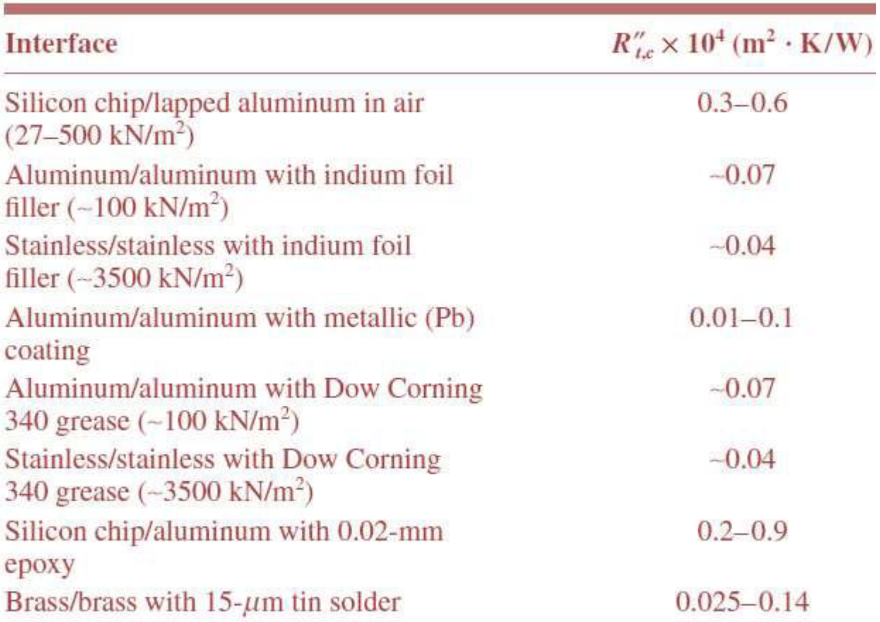
Thermal contact resistance

Thermal contact resistance for different solid/solid and solid/fluid interfaces



Thermal Resistance, $R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)

(a) Vacuum Interface			(b) Interfacial Fluid	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265



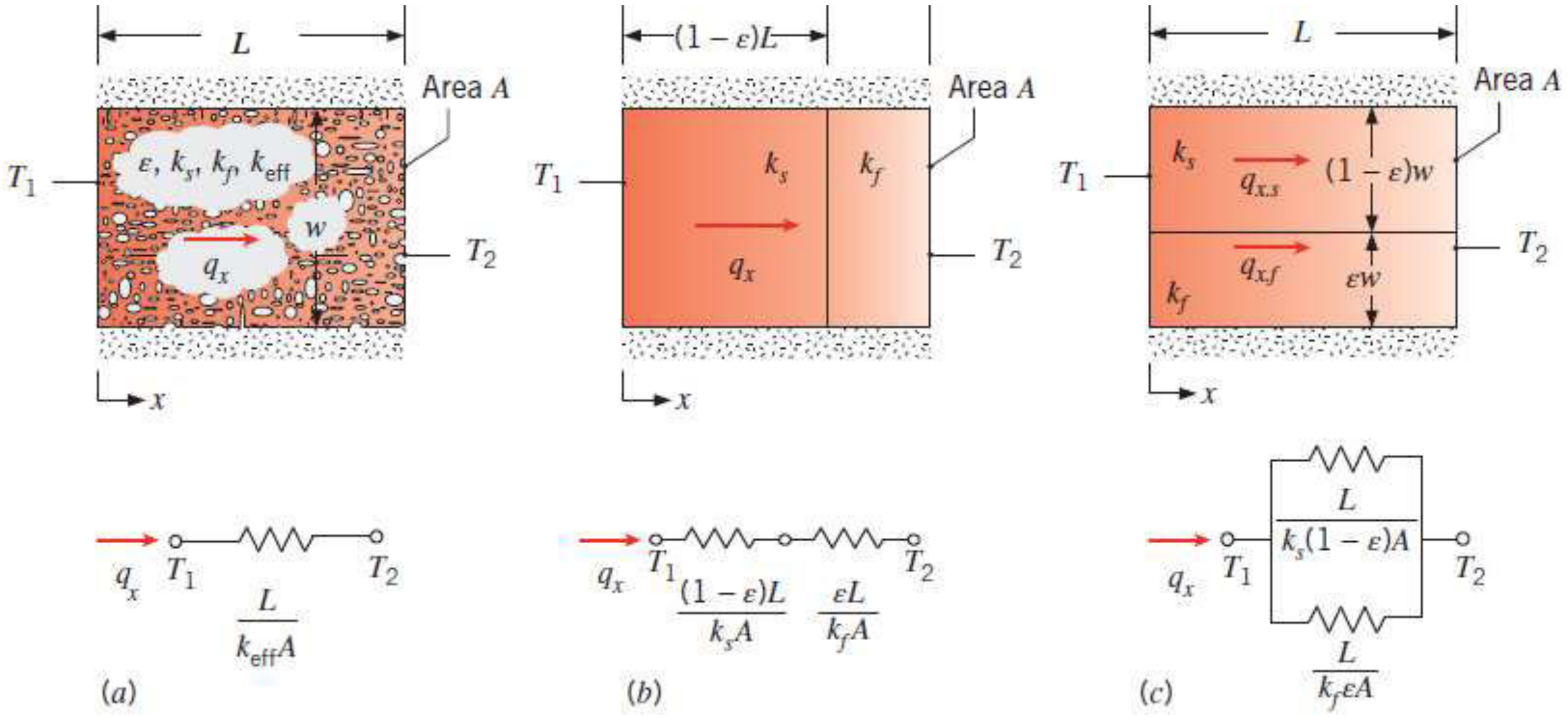
Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15-μm tin solder	0.025–0.14

Many applications involve contact between similar solids and wide range of possible filler materials. Two class of materials that are well suited to enhance interfacial k are soft metals and thermal greases. Silicon based thermal greases are attractive as they have k fifty times that of air

Thermal conductivity – Porous media

- Porous media are combinations of stationary solid and fluid
- If the fluid is either gas or liquid the resulting porous medium is said to be saturated
- In an unsaturated medium all the three phases coexist

Ex: Insulation systems, nanofluids, powder of bed with fluid in interstitial regions

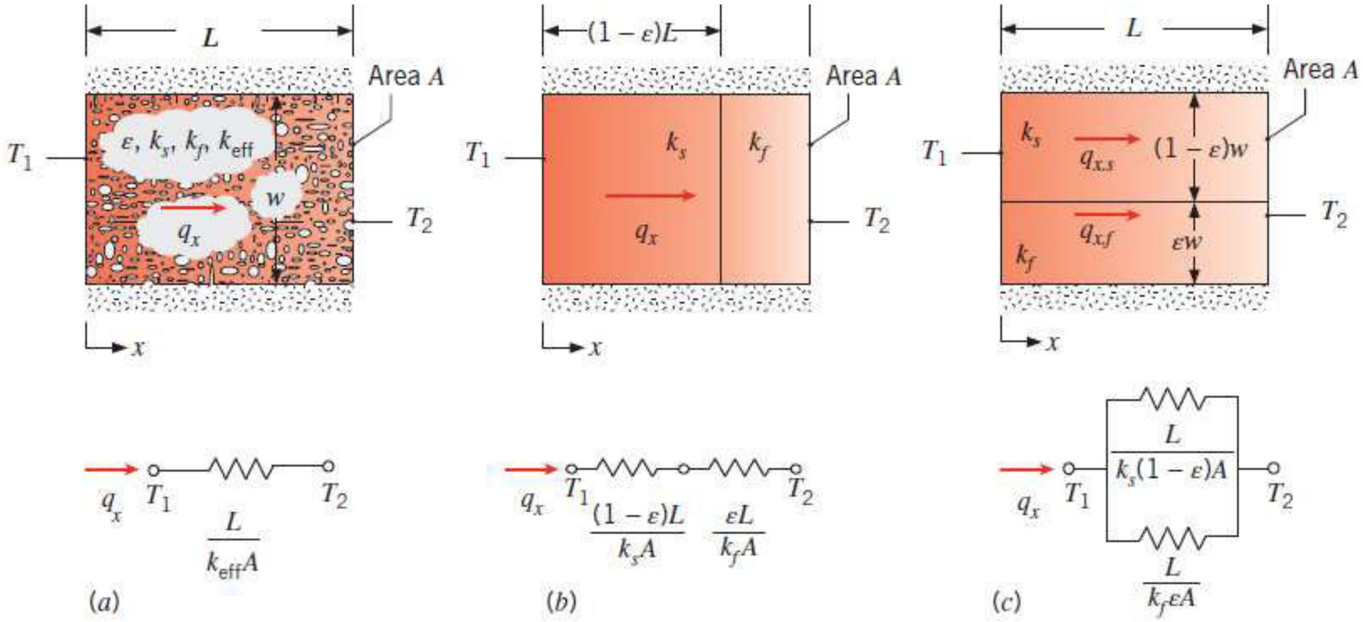


Thermal conductivity – Porous media

- For a saturated porous medium that is subjected to surface temperatures T_1 at $x = 0$ and T_2 at $x = L$, the heat rate after steady state conduction is reached is:

$$q_x = \frac{k_{\text{eff}} A}{L}(T_1 - T_2)$$

- The above equation is valid if fluid motion and radiation heat transfer are negligible
- k_{eff} – effective thermal depends on following factors:- porosity or void fraction, ε, k_{solid}, k_{fluid}, size distribution and packing arrangement of individual powder particles, contact resistance at interfaces of particles, nanoscale phenomenon



Thermal conductivity – Porous media

- If the medium is modeled as a series composite consisting of fluid of length εL and solid of length $(1 - \varepsilon)L$

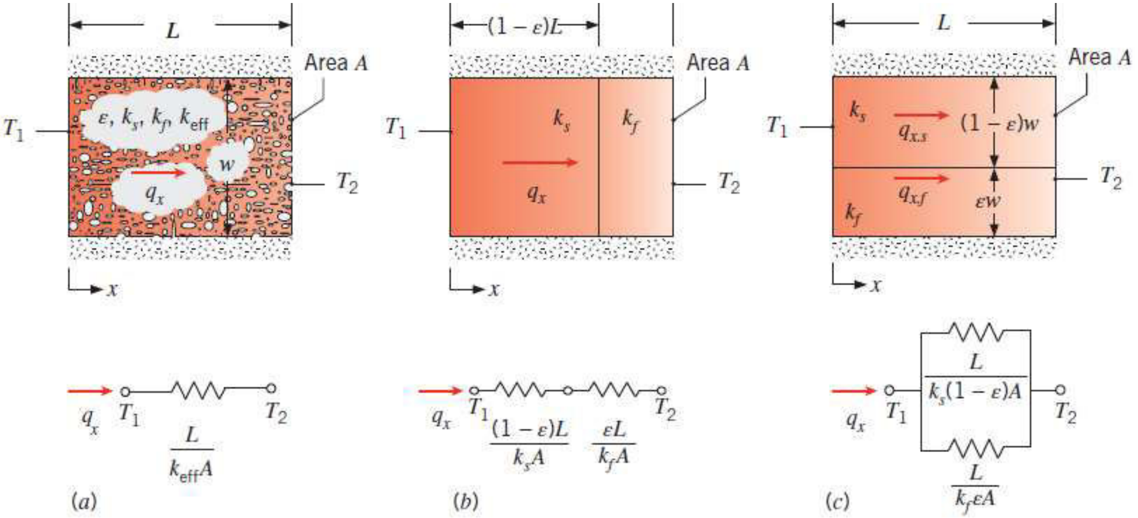
$$q_x = \frac{A\,\Delta T}{(1-\varepsilon)L/k_s + \varepsilon L/k_f}$$

→

$$q_x = \frac{k_{\text{eff}} A}{L}(T_1 - T_2)$$

- On equating left hand side and right hand side we get $$k_{\text{eff,min}} = \frac{1}{(1-\varepsilon)/k_s + \varepsilon/k_f}$$

- If the medium is described by an equivalent parallel composite wall consisting fluid region of width w and solid region of width $(1 - \varepsilon)w$, the equivalent thermal resistance is: $$k_{\text{eff,max}} = \varepsilon k_f + (1-\varepsilon)k_s$$



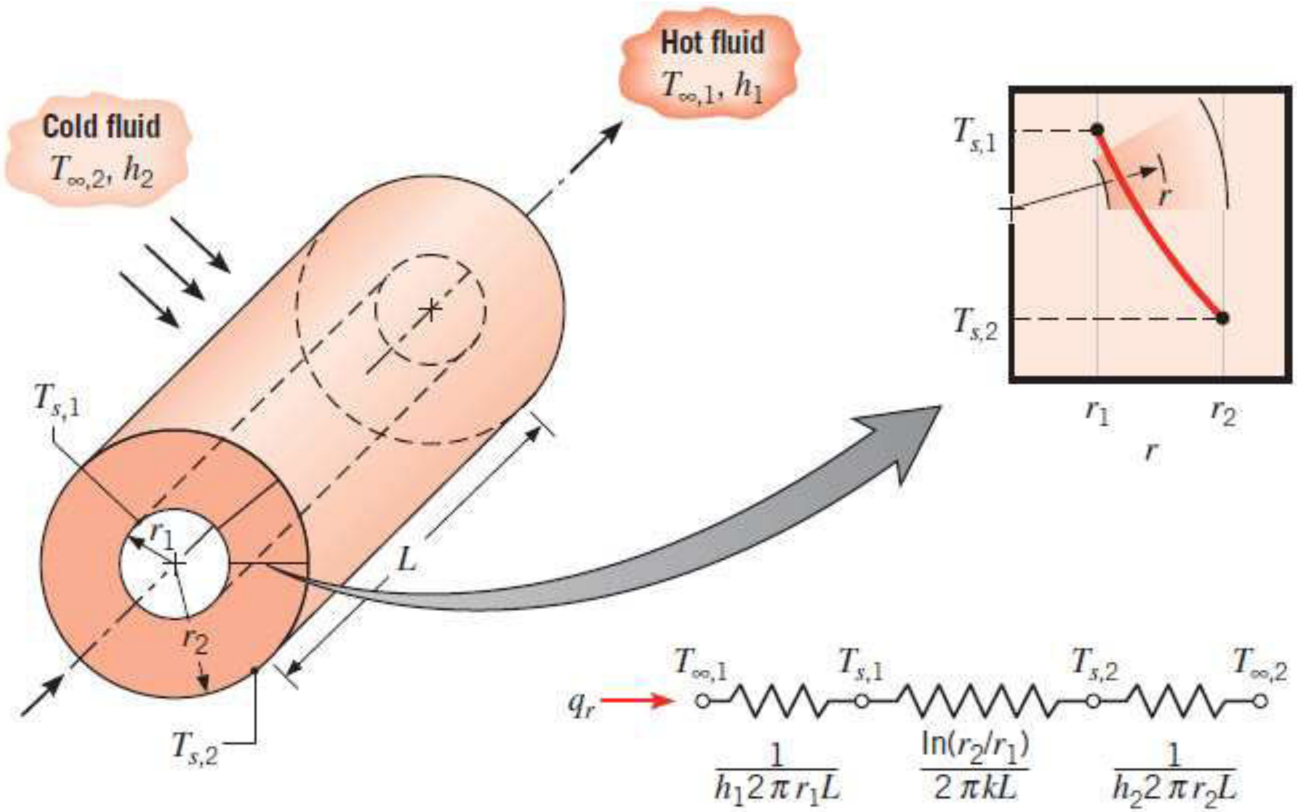
1D conduction in radial systems – Cylinder

- *Cylindrical and spherical systems often experience temperature gradients in the radial direction only and may therefore be treated as 1D*
- *Recall that the general form of heat equation for cylindrical coordinates in 3D is :*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

- *For steady state 1D case with no heat generation and storage the equation reduces to:*

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0$$



Note that k is considered variable in 1D expression of cylinder

1D conduction in radial systems – Cylinder

- The rate at which energy is conducted across cylindrical surface is expressed as:

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$

- A = 2πrL is the area normal to direction of heat transfer. Since $kr\frac{dT}{dr}$ is independent of r, the conduction heat transfer rate q_r and (not heat flux q_r'') is constant in radial direction

We determine temp distribution in cylinder by solving equation:

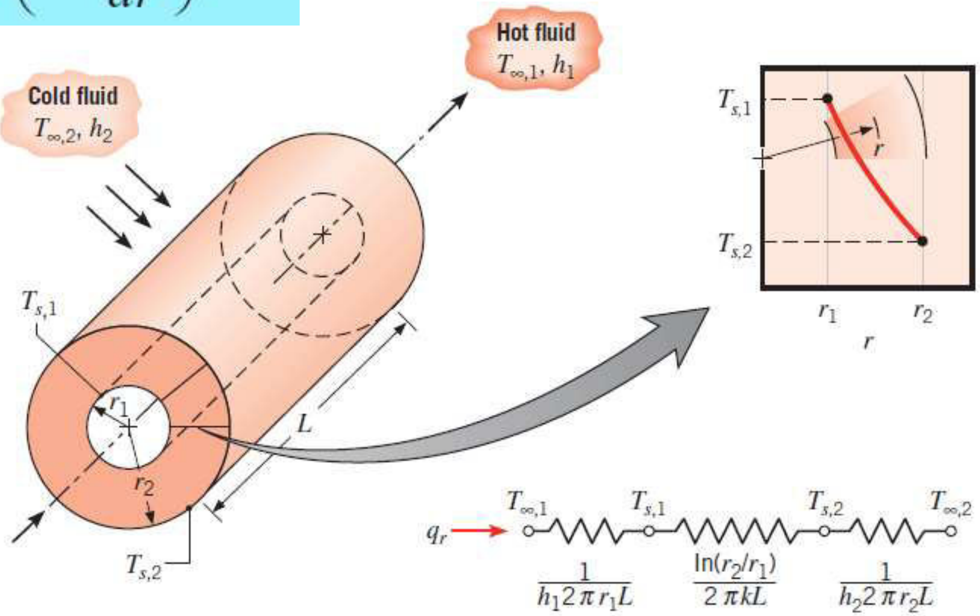
$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

$$T(r) = C_1 \ln r + C_2$$

To solve this equation boundary conditions are:

$$T(r_1) = T_{s,1} \quad \text{and} \quad T(r_2) = T_{s,2}$$

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$



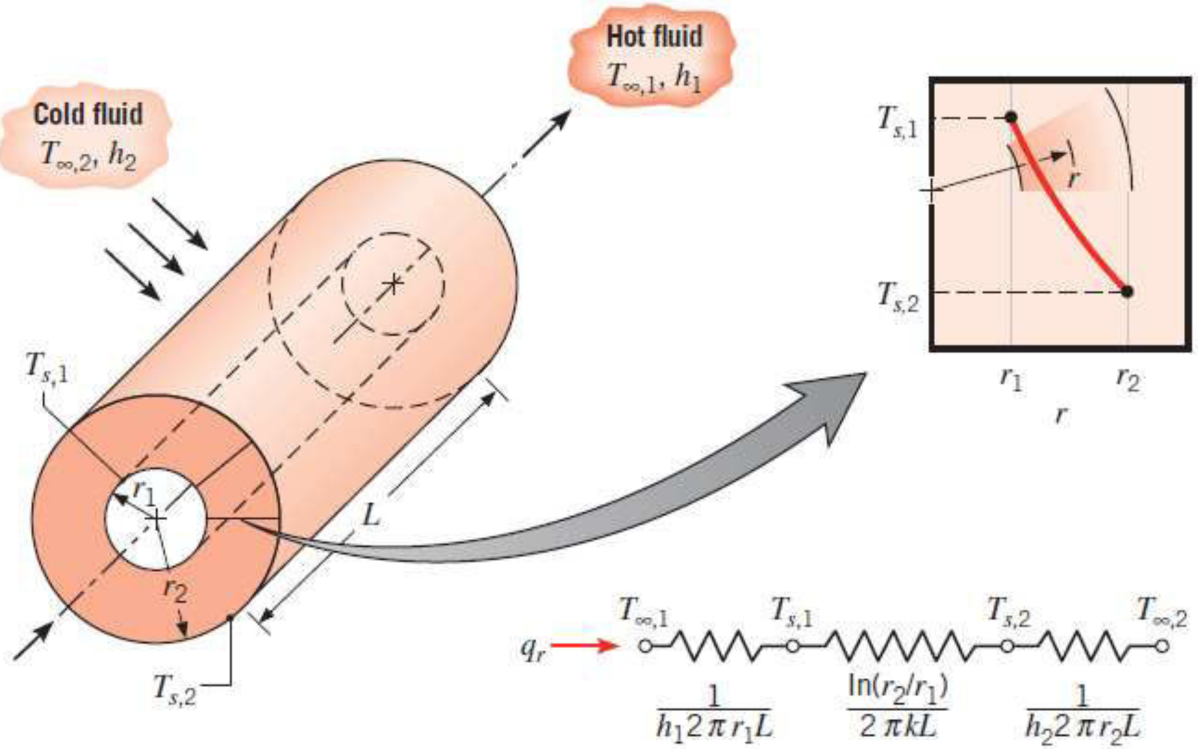
1D conduction in radial systems – Cylinder

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

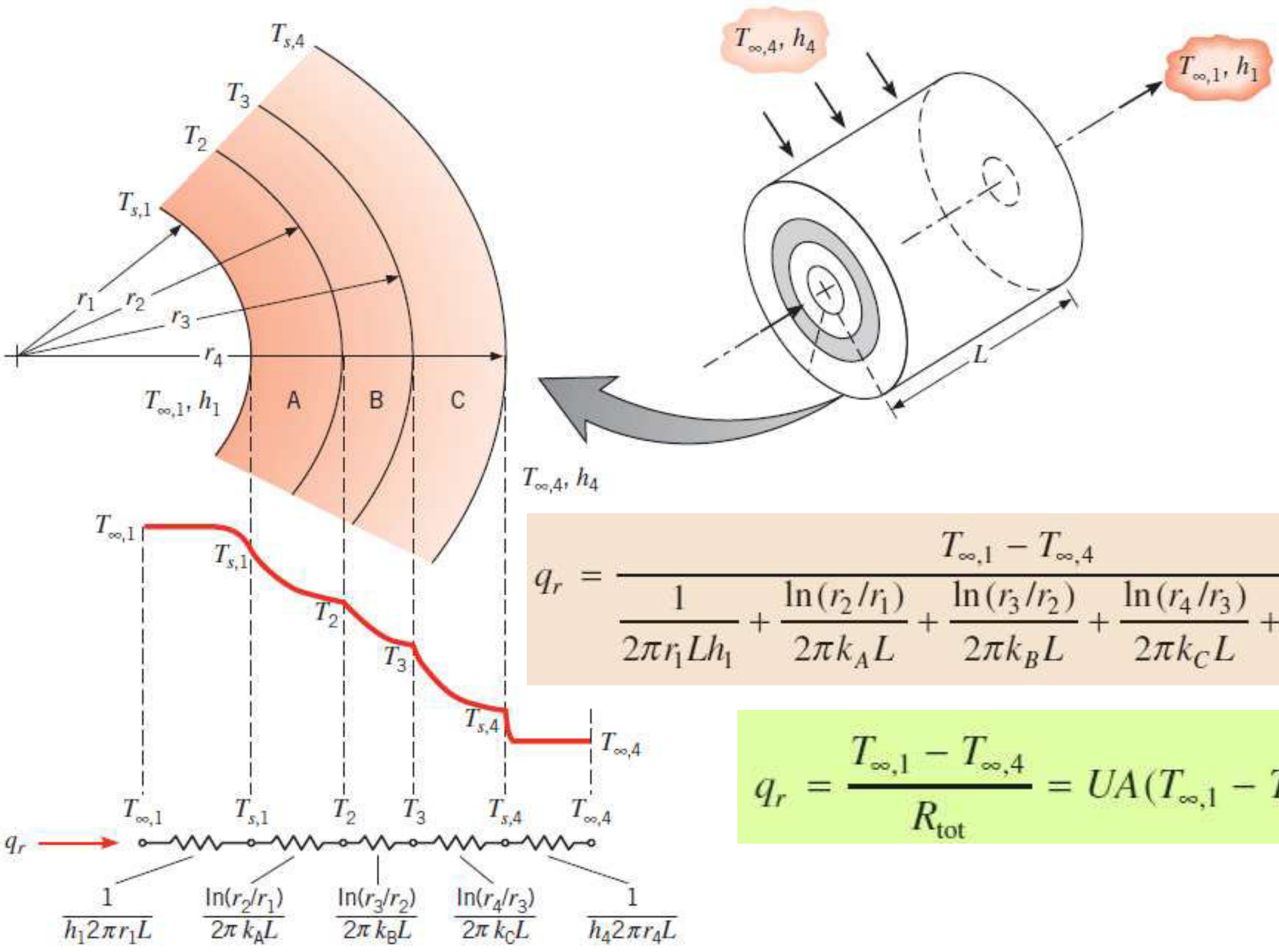
- If the above temp distribution is now used with Fourier's law:

$$q_r = \frac{2\pi L k (T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}$$

$$R_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi L k}$$



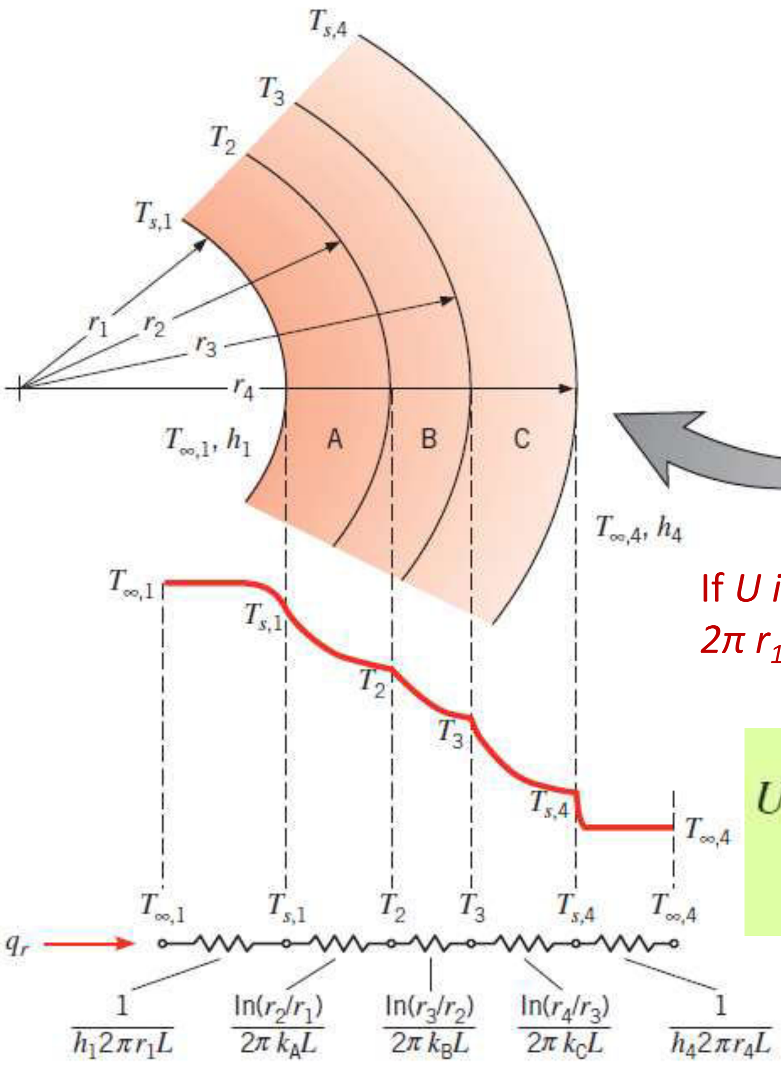
1D conduction in radial systems – Cylinder

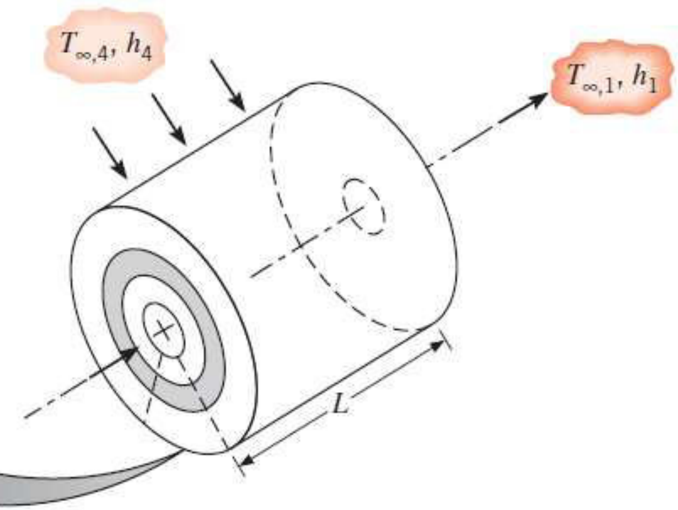


$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln(r_2/r_1)}{2\pi k_A L} + \dfrac{\ln(r_3/r_2)}{2\pi k_B L} + \dfrac{\ln(r_4/r_3)}{2\pi k_C L} + \dfrac{1}{2\pi r_4 L h_4}}$$

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{tot}}} = UA(T_{\infty,1} - T_{\infty,4})$$

1D conduction in radial systems – Cylinder





If *U is defined in terms of the arbitrary inside area, $A_1 = 2\pi r_1 L$*

$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\frac{r_2}{r_1} + \frac{r_1}{k_B}\ln\frac{r_3}{r_2} + \frac{r_1}{k_C}\ln\frac{r_4}{r_3} + \frac{r_1}{r_4}\frac{1}{h_4}}$$

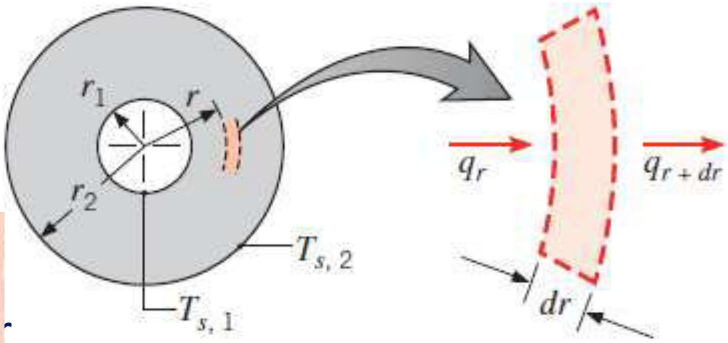
$$U_1A_1 = U_2A_2 = U_3A_3 = U_4A_4 = (\Sigma R_t)^{-1}$$

1D conduction in radial systems – Sphere

For the differential control volume, energy conservation requires that $q_r = q_{r+dr}$

For steady state , 1D conditions with no heat generation the Fourier law:

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$



A= $4\pi r^2$ is the area normal to d

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

For radial systems, q_r is consta

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)\,dT$$

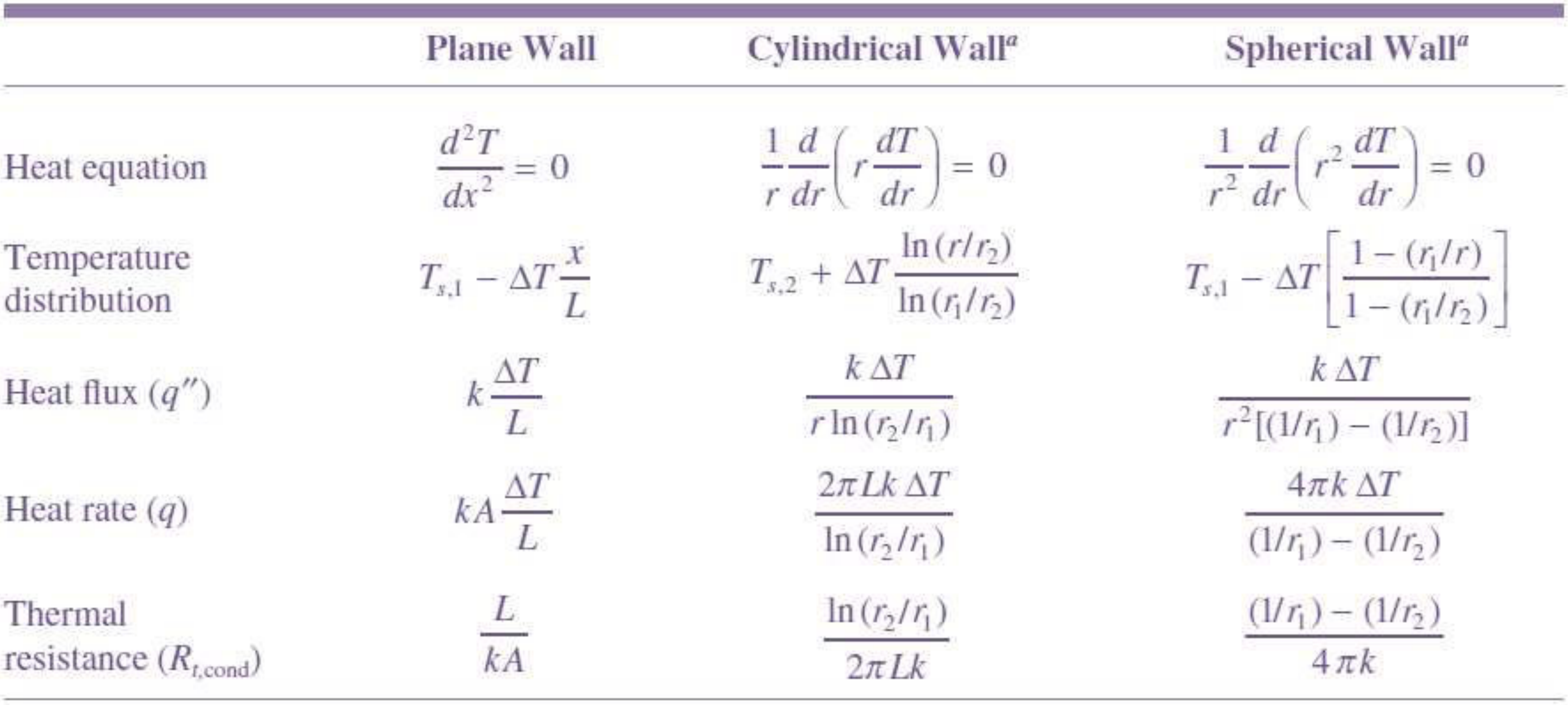
Assuming constant thermal conductivity, k, the numerical expression for q_r :

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)}$$

Thermal resistivity can then be expressed as: ➡

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Summary of 1D steady state solutions to heat equation with no generation



	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

HEAT TRANSFER

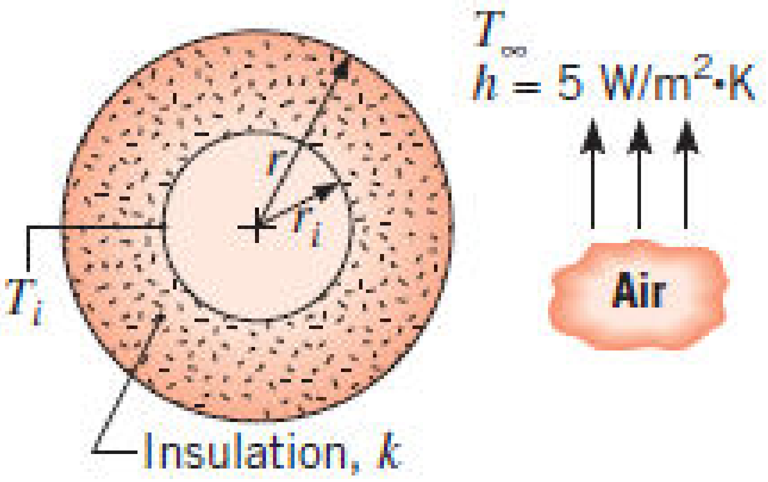
MODULE 1

CRITICAL THICKNESS OF INSULATION

Dr. Satish Idury
Associate Professor

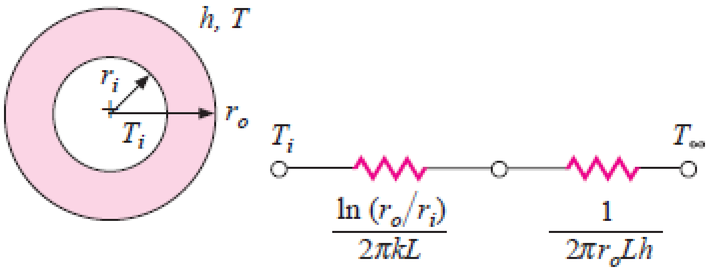
Critical thickness of insulation

- Consider a smaller diameter tube, cable or the wire whose outside surface has constant temperature and dissipates heat by convection to surroundings
- If the surface is covered with a layer of insulation it is desirable to evaluate variation in heat loss from tube surface as the thickness of insulation increases
- *As insulation is added, outer exposed surface temperature decreases due to higher conduction resistance but the surface area available for convective heat dissipation will increase*
- There are two conflicting effects leading to optimum insulation thickness



Critical thickness of insulation - Cylinder

- The inner temperature of insulation is fixed at T_i and the surface exposed to conduction environment is T_∞
- From thermal network, heat transfer is:



$$q = \frac{2\pi L\,(T_i - T_\infty)}{\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}}$$

- To determine outer radius of insulation which maximizes heat transfer: The maximization condition is:

$$\frac{dq}{dr_o} = 0 = \frac{-2\pi L\,(T_i - T_\infty)\left(\dfrac{1}{kr_o} - \dfrac{1}{hr_o^2}\right)}{\left[\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}\right]^2}$$

- The above maxima condition gives the result:

$$r_o = \frac{k}{h}$$

- If the outer radius is less than critical value, heat transfer will increase by adding more insulation.
- For outer radii greater than critical value increase in insulation thickness will cause a decrease in heat transfer

Critical thickness of insulation - Sphere

The total thermal resistance for spherical wall of outer radius , r is:

$$R_{th} = \frac{r_0 - r_i}{4\pi k r_o r_i} + \frac{1}{4\pi r_o^{2} h_0} \qquad = \frac{1}{4\pi}\left[\left\{\frac{1}{r_i} - \frac{1}{r_o}\right\}\right]\frac{1}{k_i} + \frac{1}{h_o r_o^{2}}$$

For resistance R_{th} to be minimum and Q to be maximum:

$$\frac{dR_{th}}{dr_o} = \frac{1}{4\pi}\left[\frac{1}{r_0^{2}\, ki} - \frac{2}{h_0 r_o^{3}}\right] = 0$$

$$(r_0)_{critical} \text{ for sphere} = \frac{2k_i}{h_0}$$

HEAT TRANSFER

MODULE II
1D TRANSIENT HEAT CONDUCTION

Part 2

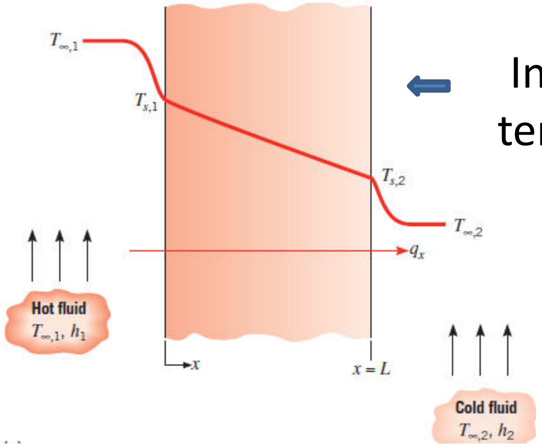
Dr. Satish Idury
Associate Professor

Transient Conduction

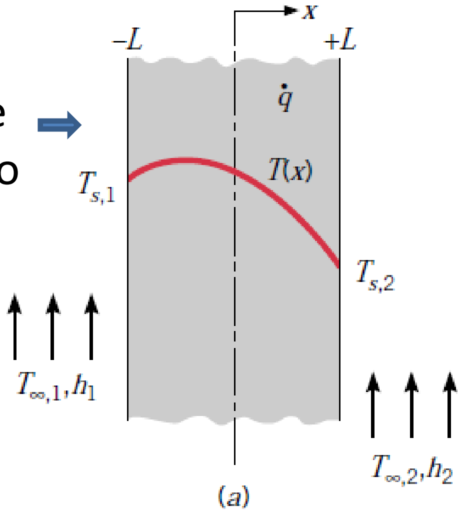
Basic heat equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

- *Heat transfer problems are time dependent*
- *Unsteady, or transient, problems typically arise when the boundary conditions of a system are changed*
- *Consider a hot metal billet that is removed from a furnace and exposed to a cool airstream*
- *Energy is transferred by conduction, convection and radiation to the surroundings until the temperature at each point in billet decreases and reaches steady state*

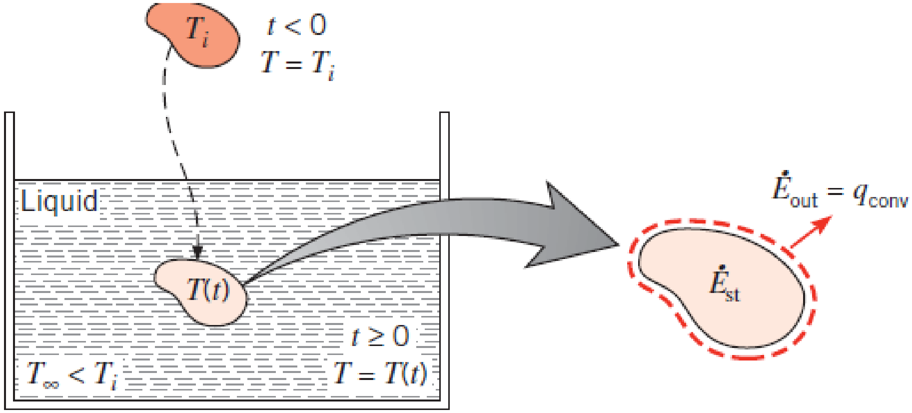


In 1D transient problems these temperature distributions are no longer valid



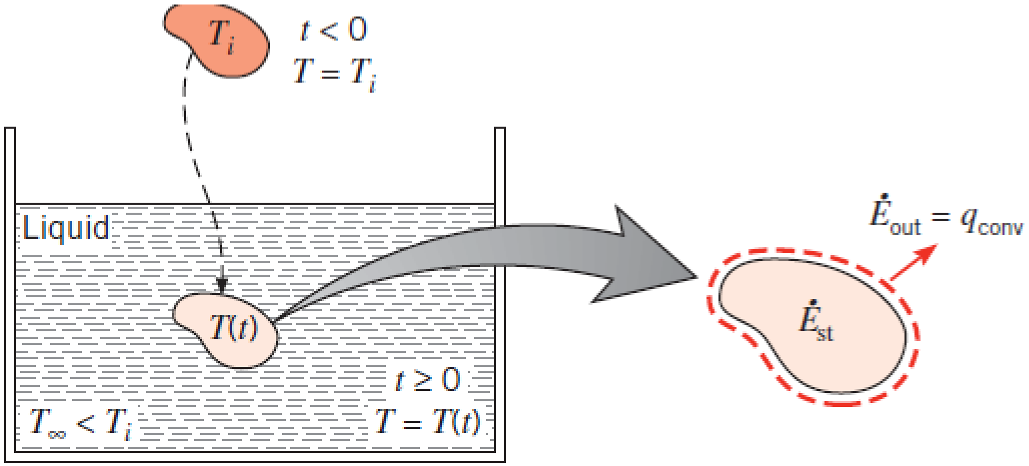
Transient Conduction

- Procedures for determining the time dependence of the temperature distribution within a solid during a transient process depend upon assumptions made upon the process
- If temperature gradients within the solid may be neglected, a comparatively simple approach, termed the lumped capacitance method, may be used to determine the variation of temperature with time
- Under conditions for which temperature gradients are not negligible, but heat transfer within the solid is one-dimensional, exact solutions to the heat equation may be used to compute the dependence of temperature on both location and time
- Systems that display limited internal thermal resistance are of interest in this module



Lumped Capacitance method

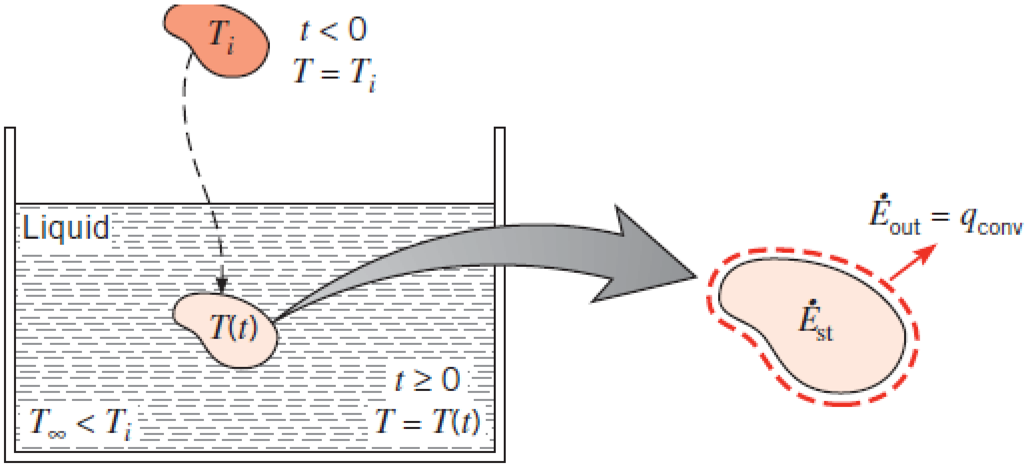
- Consider a hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$
- If the quenching is said to begin at time $t = 0$, the temperature of the solid will decrease for time $t > 0$, until it eventually reaches T_∞ due to convection heat transfer at solid – liquid interface
- Lumped capacitance method assumes that temperature of solid is spatially uniform at any instant during transient process
- Absence of thermal gradients implies infinite thermal conductivity or negligible internal resistance
- This approximation is feasible if conduction resistance is small compared to convection resistance between solid and surroundings



Lumped Capacitance method

- In neglecting temperature gradients within the solid, we can no longer consider the problem from within the framework of the heat equation, since the heat equation is a differential equation governing the spatial temperature distribution within the solid

- The transient temperature response is determined by formulating an overall energy balance on the entire solid. This balance must relate the rate of heat loss at the surface to the rate of change of the internal energy



Lumped Capacitance method

Applying energy balance to control volume shown in figure:

$$-\dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

or

$$-hA_s(T - T_\infty) = \rho Vc\frac{dT}{dt}$$

Introducing the temperature difference:

$$\theta \equiv T - T_\infty$$

⇒ $(d\theta/dt) = (dT/dt)$ if T_∞ is constant

$$\frac{\rho Vc}{hA_s}\frac{d\theta}{dt} = -\theta$$

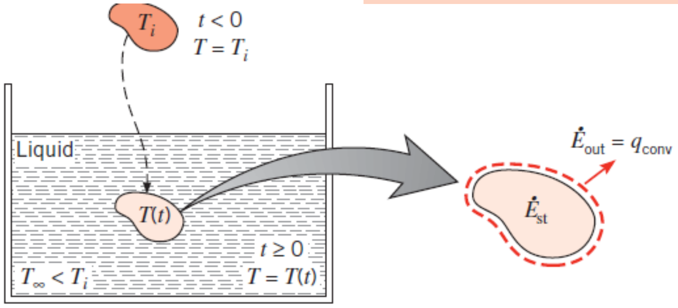
Separating variables and integrating from the initial condition, for which $t = 0$ and $T(0) = T_i$

$$\frac{\rho Vc}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt$$

$$\frac{\rho Vc}{hA_s}\ln\frac{\theta_i}{\theta} = t$$

⇒

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right]$$



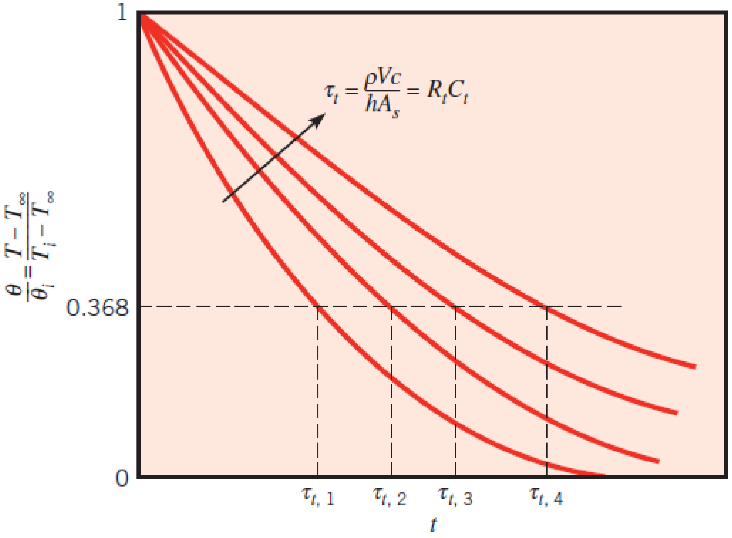
Lumped Capacitance method

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c}\right)t\right]$$

- Difference between the solid and fluid temperatures decays exponentially to zero as *t* approaches infinity
- The quantity $(\rho V_c/hA_s)$ may be interpreted as a *thermal time constant*

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho V c) = R_t C_t$$

- R_t is the resistance to convection heat transfer and C_t is the *lumped thermal capacitance* of the solid
- Any increase in R_t or C_t will cause the solid to respond more slowly to changes in its thermal environment



Lumped Capacitance method

To determine the total energy transfer Q occurring up to some time t:

$$Q = \int_0^t q\,dt = hA_s \int_0^t \theta\,dt$$

$$Q = (\rho V c)\theta_i\left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right]$$

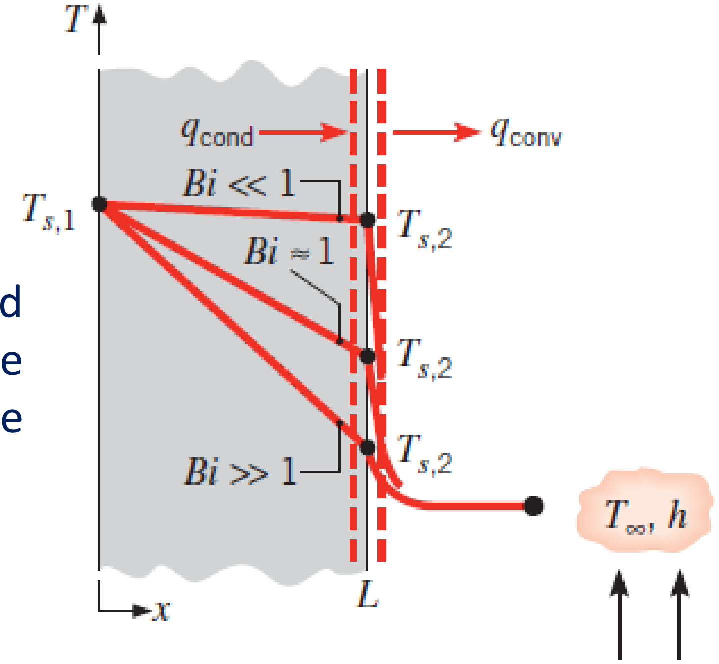
The quantity Q is related to the change in the internal energy of the solid

$$-Q = \Delta E_{st}$$

Applicability of lumped capacitance method:

Consider steady-state conduction through the plane wall of area A

One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$. The temperature of this surface will be some intermediate value $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$

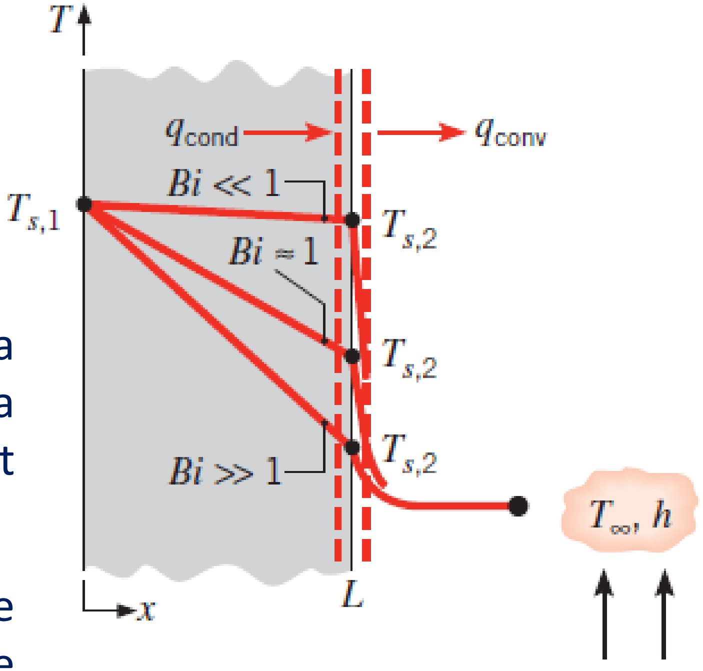


Validity of lumped Capacitance method

Applying the surface energy balance:

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_{\infty})$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_{\infty}} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,\text{cond}}}{R_{t,\text{conv}}} = \frac{hL}{k} \equiv Bi$$



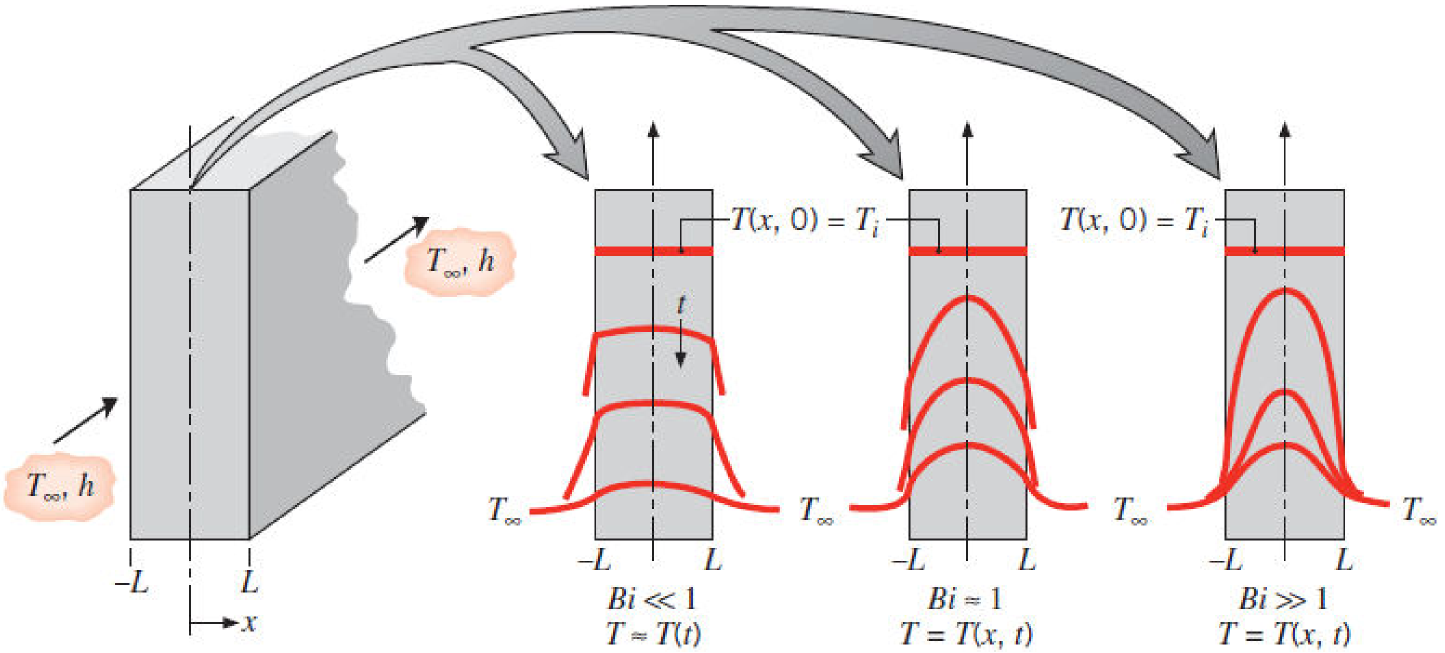
- ❖ The quantity (hL/k) appearing in above equation is a dimensionless parameter: Biot number, and plays a fundamental role in conduction problems that involve surface convection effects
- ❖ Biot number provides a measure of the temperature drop in the solid relative to the temperature difference between the solid's surface and the fluid
- ❖ *If Bi << 1, the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer*
- ❖ The assumption of a uniform temperature distribution within the solid is reasonable if the Biot number is small

Transient temperature distribution for different Biot numbers

Consider the plane wall of figure below, which is initially at a uniform temperature T_i *and experiences convection cooling when it is immersed in a fluid of* $T_\infty < T_i$

For 1D problem, temperature variation with position and time T(x, t) is a strong function of Biot number:

- *Bi* << 1 the temperature gradients in the solid are small and the assumption of a uniform temperature distribution, $T(x, t) \approx T(t)$ is reasonable
- *Bi* >> 1, the temperature difference across the solid is much larger than that between the surface and the fluid



Transient temperature distribution for different Biot numbers

$$Bi = \frac{hL_c}{k} < 0.1$$

If the above condition is satisfied the error associated with lumped capacitance is small

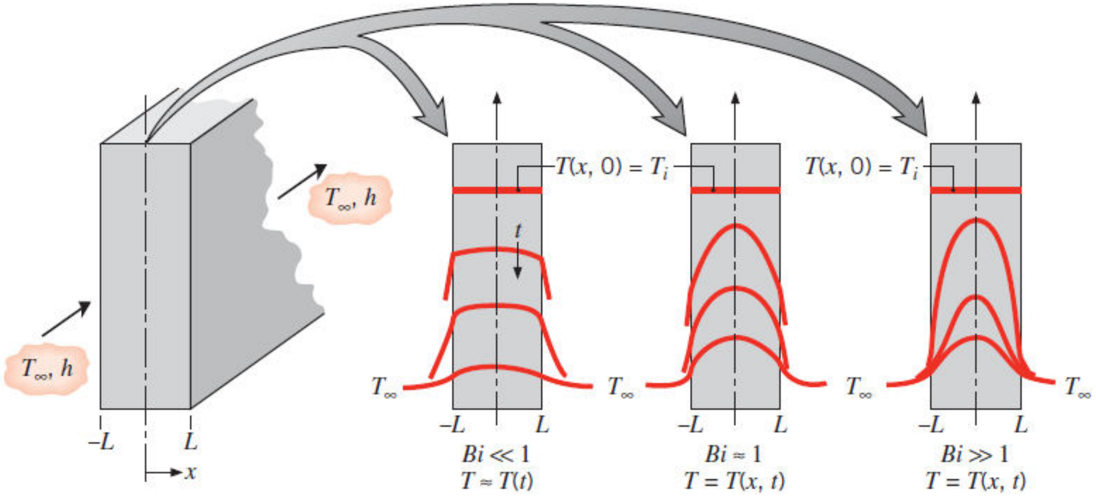
L_c is called characteristic length and is the ratio of solid's volume to surface area.

L_c = L for plane wall of thickness 2L, and $r_o/2$ for cylinder, $r_o/3$ for sphere

L_c also corresponds to the maximum spatial temperature difference

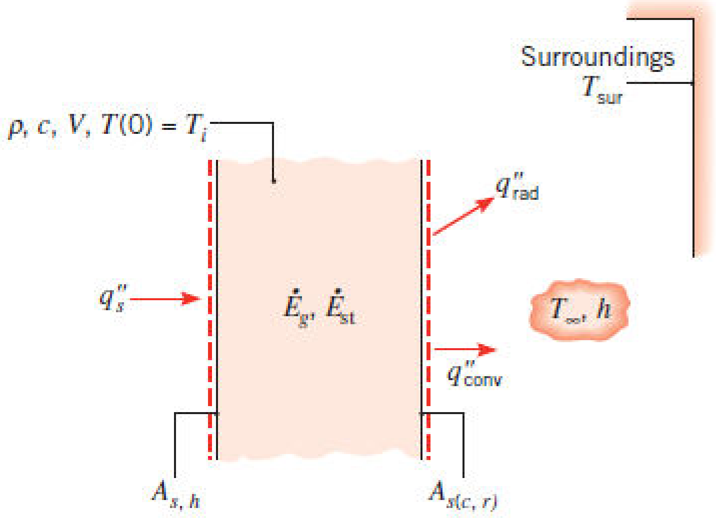
$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\,\frac{k}{\rho c}\,\frac{t}{L_c^2} = \frac{hL_c}{k}\,\frac{\alpha t}{L_c^2} \quad \Rightarrow \quad \frac{hA_s t}{\rho V c} = Bi \cdot Fo$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$$



Generalized lumped Capacitance analysis

- *In addition to convection from the adjoining fluid, radiation exchange can induce transient thermal conditions*
- *Transient conditions can be induced by applying heat flux at surfaces*
- *Similarly initiation of thermal energy generation also lead to transient conditions*
- Figure below depicts the general situation for which thermal conditions within a solid may be influenced simultaneously by convection, radiation, an applied surface heat flux, and internal energy generation
- The imposed heat flux q_s and the convection radiation heat transfer occur at mutually exclusive portions of the surface
- Though convection and radiation have been prescribed for the same surface, the surfaces may, in fact, differ ($A_{s,c} \neq A_{s,r}$)



Generalized lumped Capacitance analysis

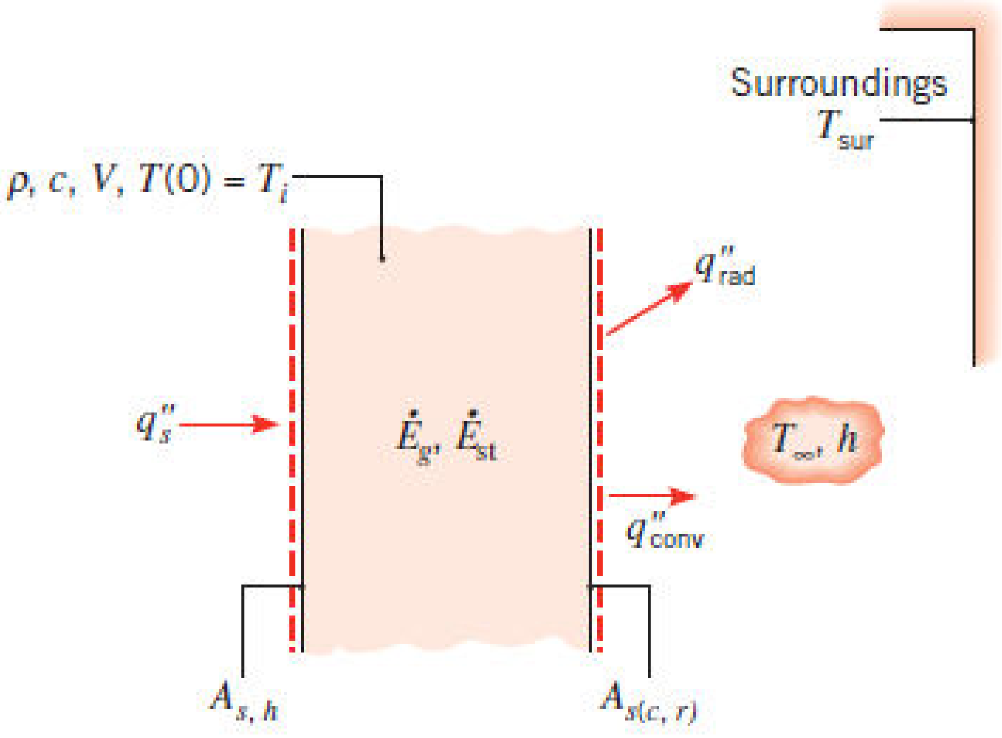
Applying conservation of energy at any instant t

$$q_s'' A_{s,h} + \dot{E}_g - (q''_{\text{conv}} + q''_{\text{rad}}) A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

or

$$q_s'' A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T^4_{\text{sur}})] A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

The above equation is a nonlinear, first-order, nonhomogeneous, ordinary differential equation that cannot be integrated to obtain an exact solution



Generalized lumped Capacitance analysis

Case A: Radiation only:

$$\cancel{q''_s A_{s,h}} + \cancel{\dot{E}_g} - [\cancel{h(T - T_\infty)} + \varepsilon\sigma(T^4 - T_{\text{sur}}^4)]A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

$$\rho V c \frac{dT}{dt} = -\varepsilon A_{s,r} \sigma (T^4 - T_{\text{sur}}^4)$$

Separating variables and integrating from the initial condition to any time t,

$$\frac{\varepsilon A_{s,r}\sigma}{\rho V c} \int_0^t dt = \int_{T_i}^{T} \frac{dT}{T_{\text{sur}}^4 - T^4}$$

Time required to reach temperature T is given by:

$$t = \frac{\rho V c}{4\varepsilon A_{s,r}\sigma T_{\text{sur}}^3} \left\{ \ln\left|\frac{T_{\text{sur}} + T}{T_{\text{sur}} - T}\right| - \ln\left|\frac{T_{\text{sur}} + T_i}{T_{\text{sur}} - T_i}\right| + 2\left[\tan^{-1}\left(\frac{T}{T_{\text{sur}}}\right) - \tan^{-1}\left(\frac{T_i}{T_{\text{sur}}}\right)\right]\right\}$$

Generalized lumped Capacitance analysis

Case B: Negligible radiation:

$$q_s'' A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \cancel{\varepsilon\sigma(T^4 - T_{\text{sur}}^4)}]A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

In the above general equation all quantities except T are independent of t:

Let $\theta \equiv T - T_\infty$, where $d\theta/dt = dT/dt$,

The equation reduces to the form : $\frac{d\theta}{dt} + a\theta - b = 0$

$$a \equiv (hA_{s,c}/\rho V c) \qquad b \equiv [(q_s'' A_{s,h} + \dot{E}_g)/\rho V c]$$

Let $\theta' \equiv \theta - \frac{b}{a}$ then $\frac{d\theta'}{dt} + a\theta' = 0$

Separating variables and integrating from 0 to *t*

$$\frac{\theta'}{\theta_i'} = \exp(-at)$$

or

$$\frac{T - T_\infty - (b/a)}{T_i - T_\infty - (b/a)} = \exp(-at)$$

Generalized lumped Capacitance analysis

Case C: Convection only with variable convection coefficient:

$$\cancel{q''_s A_{s,h}} + \cancel{\dot{E}_g} - [h(T - T_\infty) + \cancel{\varepsilon\sigma(T^4 - T^4_{\text{sur}})}]A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

In free convection or boiling, the convection coefficient h varies with the temperature difference between the object and the fluid. In such situations convection coefficient can be approximated as:

$$h = C(T - T_\infty)^n$$

Then

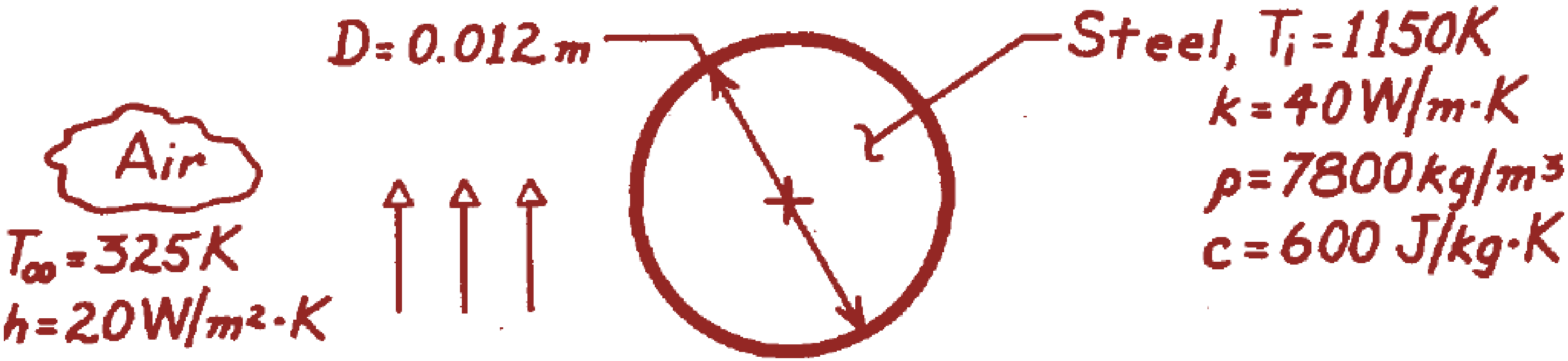
$$-C(T - T_\infty)^n A_{s,c}(T - T_\infty) = -CA_{s,c}(T - T_\infty)^{1+n} = \rho V c \frac{dT}{dt}$$

Substituting and d ϑ/dt = dT/dt into the preceding expression, separating variables and integrating gives

$$\frac{\theta}{\theta_i} = \left[\frac{nCA_{s,c}\theta_i^n}{\rho V c}t + 1\right]^{-1/n}$$

Problem 1:

Steel balls 12 mm in diameter are annealed by heating to 1150 K and then slowly cooling to 400 K in an air environment for which T_∞ = 325 K and h = 20 W/m^2 K. Assuming the properties of the steel to be k = 40 W/mK, ρ = 7800 kg/m^3, and c = 600 J/kg K, estimate the time required for the cooling process.



solution:

$$\mathrm{Bi} = \frac{hL_c}{k} = \frac{h(r_o/3)}{k} = \frac{20\ \mathrm{W/m^2 \cdot K}\ (0.002\mathrm{m})}{40\ \mathrm{W/m \cdot K}} = 0.001.$$

As Biot number is << 0.1, lumped capacitance method can be applied for transient conduction analysis

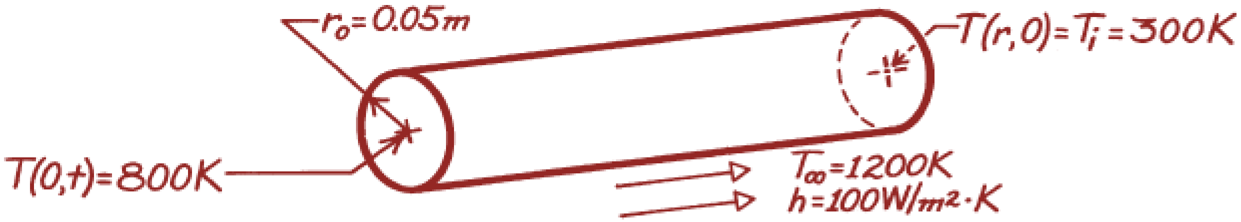
$$t = \frac{\rho V c_p}{hA_s} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho\left(\pi D^3/6\right)c_p}{h\pi\ D^2} \ln \frac{T_i - T_\infty}{T - T_\infty}$$

$$t = \frac{7800\mathrm{kg/m^3}\,(0.012\mathrm{m})\,600\mathrm{J/kg \cdot K}}{6 \times 20\ \mathrm{W/m^2 \cdot K}} \ln \frac{1150 - 325}{400 - 325}$$

$$t = 1122\ \mathrm{s}\ = 0.312\mathrm{h}$$

Problem 2:

Carbon steel (AISI 1010) shafts of 0.1 m diameter are heat treated in a gas-fired furnace whose gases are at 1200 K and provide a convection coefficient of 100 W/m^2 K. If the shafts enter the furnace at 300 K, how long must they remain in the furnace to achieve a centerline temperature of 800 K?



ρ= 7832 kg / m^3, k = 51.2 W/m.K, c= 541 J/Kg K, α = 1.21x 10^5 m^2/s

Solution:

First calculate Biot number to check applicability of lumped capacitance method:

$$\mathrm{Bi} = \frac{hr_o/2}{k} = \frac{100\ \mathrm{W/m^2 \cdot K}(0.05\ \mathrm{m}/2)}{51.2\ \mathrm{W/m \cdot K}} = 0.0488.$$

As Biot number is << 0.1, lumped capacitance method can be applied

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hAs}{\rho Vc}\right)t\right] = \exp\left[-\frac{4h}{\rho cD}t\right]$$

$$\ln\left(\frac{800-1200}{300-1200}\right) = -0.811 = -\frac{4 \times 100\ \mathrm{W/m^2 \cdot K}}{7832\ \mathrm{kg/m^3}\,(541\ \mathrm{J/kg \cdot K})\,0.1\ \mathrm{m}}t$$

$$t = 859\ \mathrm{s}.$$

HEAT TRANSFER - 80329

MODULE III

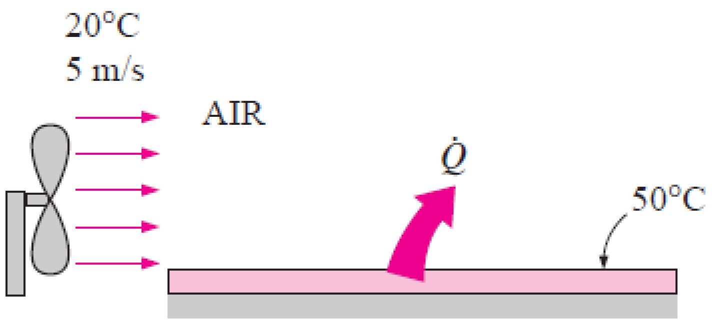
FUNDAMENTALS OF CONVECTIVE HEAT TRANSFER

Part 1

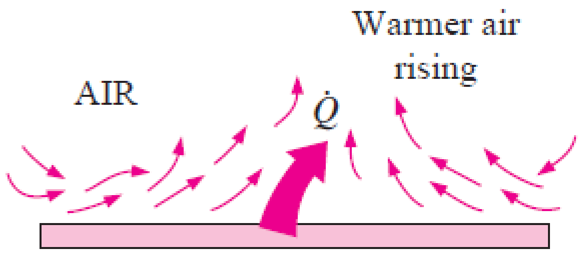
Dr. Satish Idury
Associate Professor

Physical mechanism of Convection

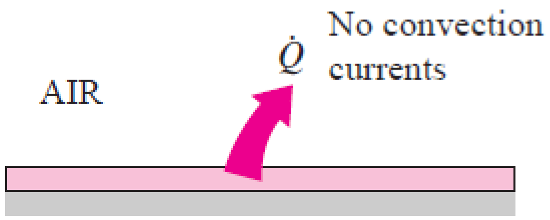
- Heat transfer through a liquid or gas, can be by conduction or convection, depending on the presence of any bulk fluid motion
- Conduction in a fluid can be viewed as the limiting case of convection, corresponding to the case of quiescent fluid
- *Convection heat transfer is complicated by the fact that it involves fluid motion as well as heat conduction*
- Fluid motion enhances heat transfer, since it brings hotter and cooler chunks of fluid into contact, initiating higher rates of conduction at a greater number of sites in a fluid



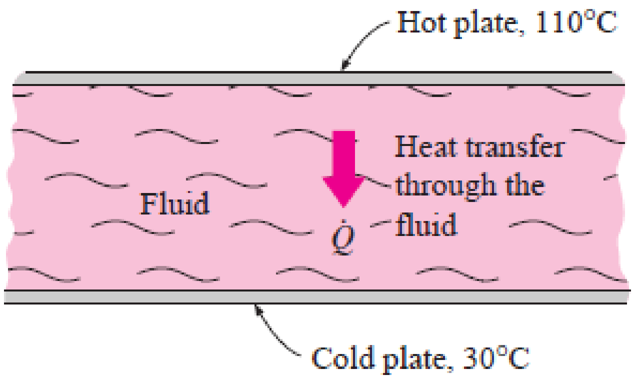
(*a*) Forced convection



(*b*) Free convection

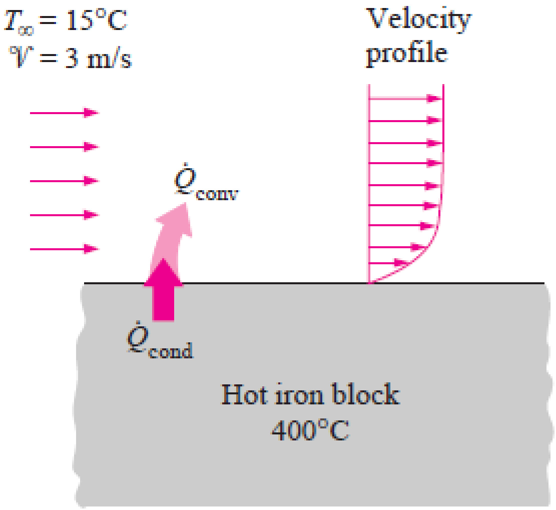


(*c*) Conduction



Physical mechanism of Convection

- Hot iron block will cool faster if the fan is switched to a higher speed. Replacing air by water will enhance the convection heat transfer even more
- Convection heat transfer depends on the fluid properties : *dynamic viscosity μ*, *thermal conductivity k*, *density ρ*, and *specific heat C_p*, as well as the *fluid velocity*
- Convection heat transfer also depends on geometry and roughness of solid surfaces and whether the flow is laminar or turbulent



The cooling of a hot block by forced convection

Convection is the most complex mechanism of heat transfer.

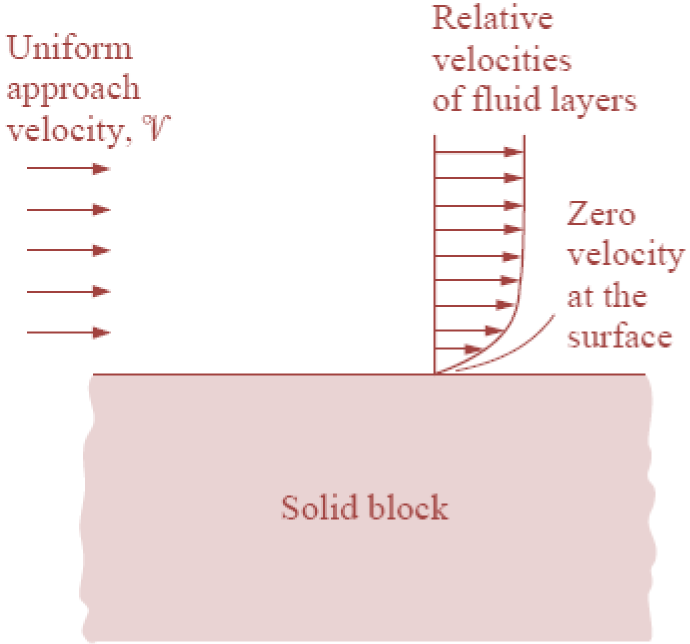
The rate of convection heat transfer is observed to be proportional to the temperature difference and is conveniently expressed by ***Newton's law of cooling***

$$\dot{q}_{\text{conv}} = h(T_s - T_\infty) \qquad (\text{W/m}^2)$$

No slip and No temperature jump condition

- *The fluid layer in direct contact with a solid surface "sticks" to the surface and there is no slip. In fluid flow, this phenomenon is known as the* ***no-slip condition,*** *and it is due to the viscosity of the fluid*

No-slip condition leads to velocity profile as shown



- *For two bodies at different temperatures in contact, heat transfer occurs until equilibrium temperature at the point of contact. A fluid and a solid surface having same temperature at the point of contact is known as* ***no-temperature-jump condition***

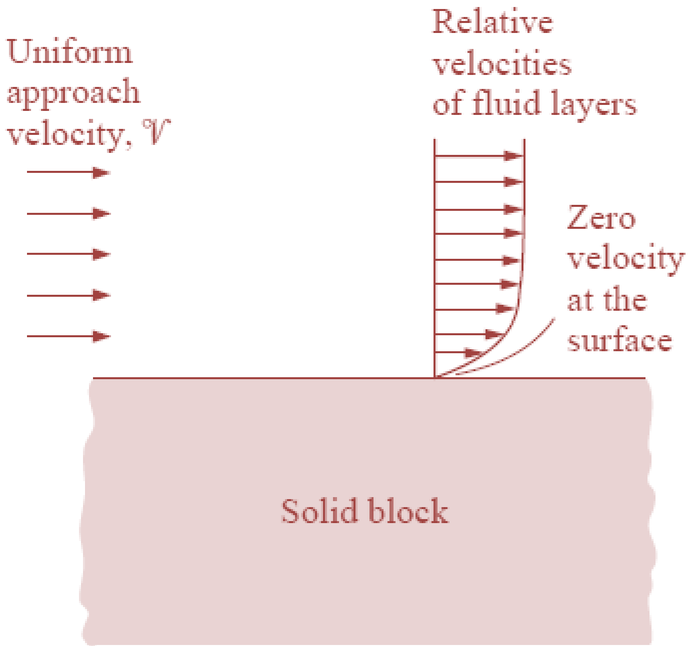
Heat transfer from the solid surface to the fluid layer adjacent to the surface is by pure conduction

$$\dot{q}_{\text{conv}} = \dot{q}_{\text{cond}} = -k_{\text{fluid}} \left.\frac{\partial T}{\partial y}\right|_{y=0} \qquad (\text{W/m}^2)$$

Classification of flows

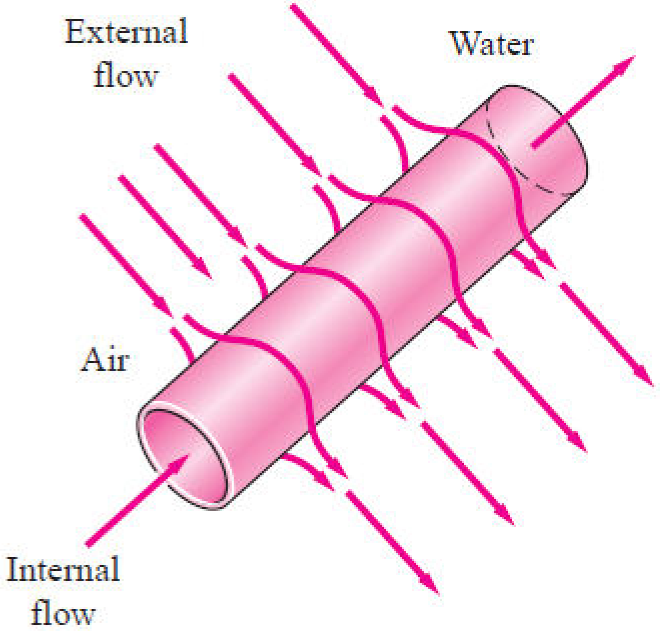
Viscous versus Inviscid flow

- Internal resistance to flow is called the **viscosity,** which is a measure of internal stickiness of the fluid. Viscosity is caused by cohesive forces between the molecules in liquids, and by the molecular collisions in gases. In viscous flows the effects of viscosity are significant while in Inviscid flows viscous effects are negligible



Internal versus external flow

- The flow of an unbounded fluid over a surface such as a plate, a wire, or a pipe is **external flow.** The flow in a pipe or duct is **internal flow** if the fluid is completely bounded by solid surfaces



Classification of flows

Compressible versus incompressible flow

- Mach number (Ma) calculation denotes whether a flow is compressible or incompressible. In the expression below , *V* is flow velocity and *a* is speed of the sound of a fluid

$$\mathrm{Ma} = \frac{V}{a}$$

For incompressible flow: ➡ $\mathrm{Ma} = \frac{V}{a} \ll 1$

- Under small Mach number conditions, changes in fluid density are everywhere small in the flow field. Incompressible flow requires only momentum and continuity analysis

For compressible flow: ➡ $\mathrm{Ma} = \frac{V}{a} > 0.3$

- If the density change is significant, it follows from the equation of state that the temperature and pressure changes are also substantial. Large temperature changes imply that the energy equation can no longer be neglected

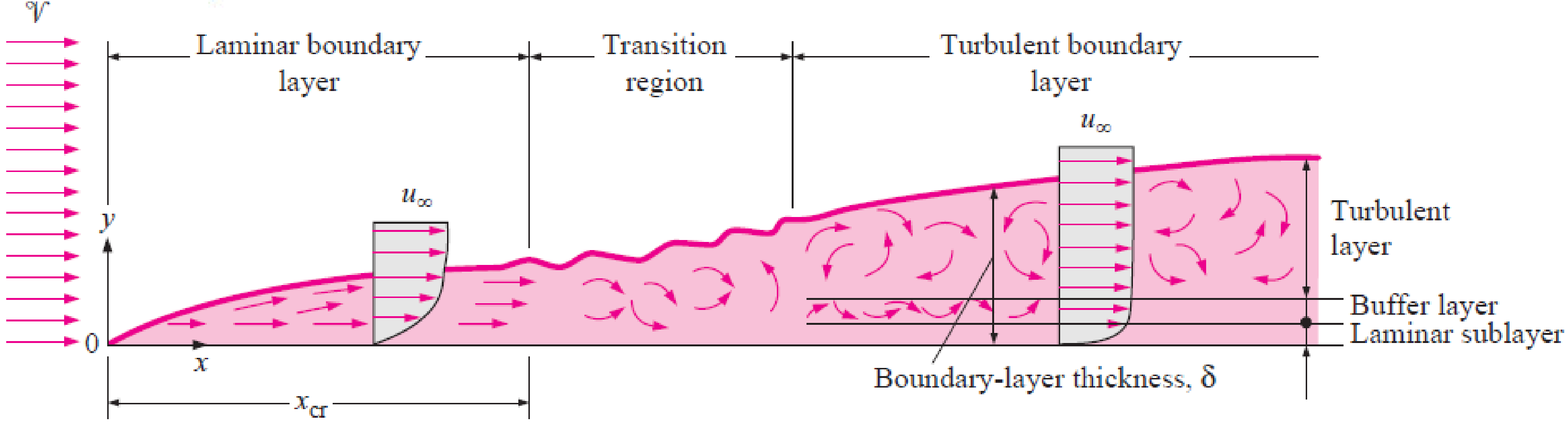
Classification of flows

Laminar versus turbulent flow

- The highly ordered fluid motion characterized by smooth streamlines is called **laminar.** The flow of high-viscosity fluids such as oils at low velocities is typically laminar
- The highly disordered fluid motion that typically occurs at high velocities characterized by velocity fluctuations is called **turbulent.** The flow of low-viscosity fluids such as air at high velocities is typically turbulent
- The flow regime greatly influences the heat transfer rates and the required power required for pumping the fluid

$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

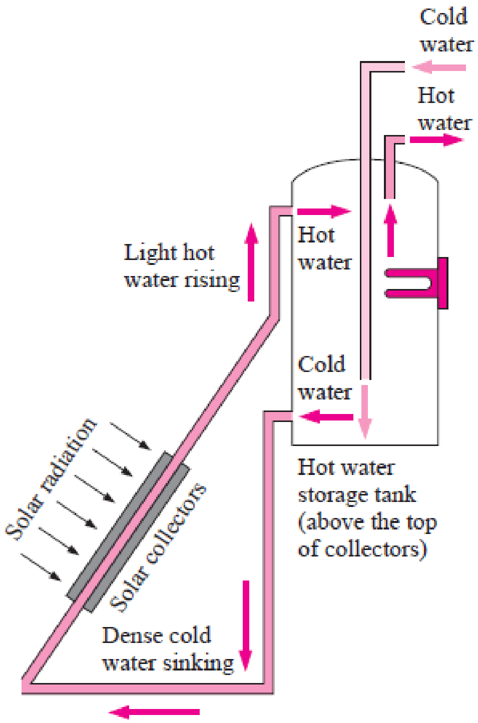
Reynolds number characterizes whether flow is laminar or turbulent



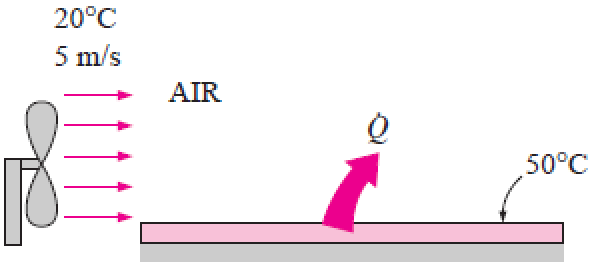
Classification of flows

Natural versus Forced flow

- In **natural flows,** any fluid motion is due to a natural means such as the buoyancy effect, which manifests itself as the rise of the warmer (and thus lighter) fluid and the fall of cooler (and thus denser) fluid
- In **forced flow,** a fluid is forced to flow over a surface or in a pipe by external means such as a pump or a fan



Natural flow



Forced flow

Classification of flows

Steady versus Unsteady (Transient) flow:

- During steady flow, the fluid properties can change from point to point within a device, but at any fixed point they remain constant. Many devices such as turbines, compressors, boilers, condensers, and heat exchangers operate for long periods of time under the same conditions, and they are classified as *steady-flow devices. In transient flow conditions the properties of a fluid at a fixed condition vary with time*

1D, 2D and 3D flows:

- A flow field is best characterized by the velocity distribution, and thus a flow is said to be one, two-, or three-dimensional if the flow velocity varies in one, two, or three primary dimensions, respectively

- A typical fluid flow involves a three-dimensional geometry and the velocity may vary in all three dimensions rendering the flow three-dimensional [(x, y, z) in rectangular or (r, ϑ , z) in cylindrical coordinates

HEAT TRANSFER - 80329

MODULE III

FUNDAMENTALS OF CONVECTIVE HEAT TRANSFER

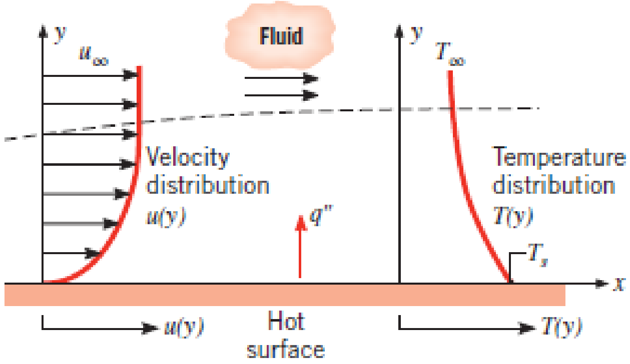
Concepts of Dimensional analysis and similarity

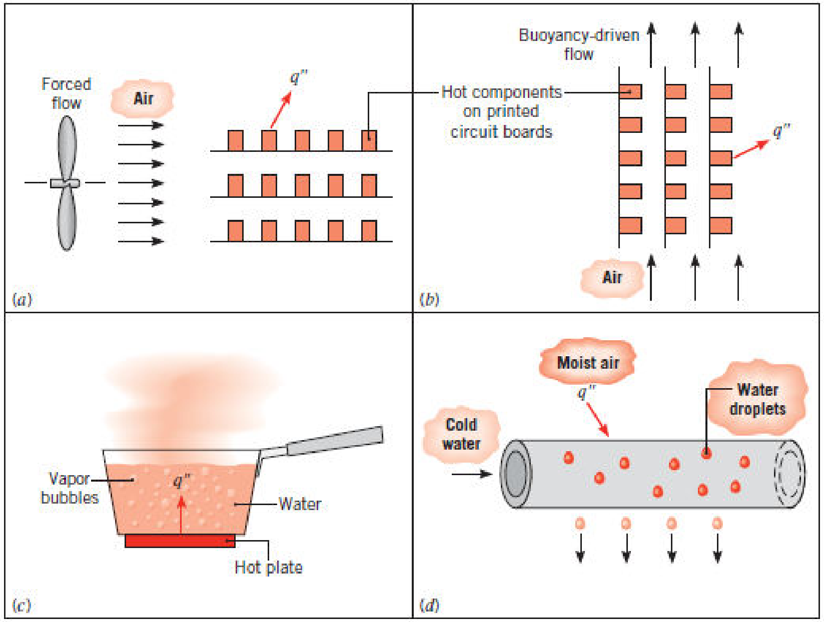
Part 2

Dr. Satish Idury
Associate Professor

Heat rate equation for convection – Newton's law of cooling

- Convection heat transfer mode comprise of two mechanisms: random molecular motion (diffusion) and bulk/macroscopic motion of fluid
- Convection is referred to cumulative transport and advection refers to bulk fluid motion alone
- Fluid - surface interaction results in the formation of hydrodynamic boundary layer
- In case of thermal gradient between surface and flow temperature, thermal boundary layer is developed
- Boundary layer phenomenon governs an important role in convective heat transfer mode
- In convection mode, sensible heat and latent heat exchange are feasible
- **Convective heat transfer coefficient depends upon boundary layer**

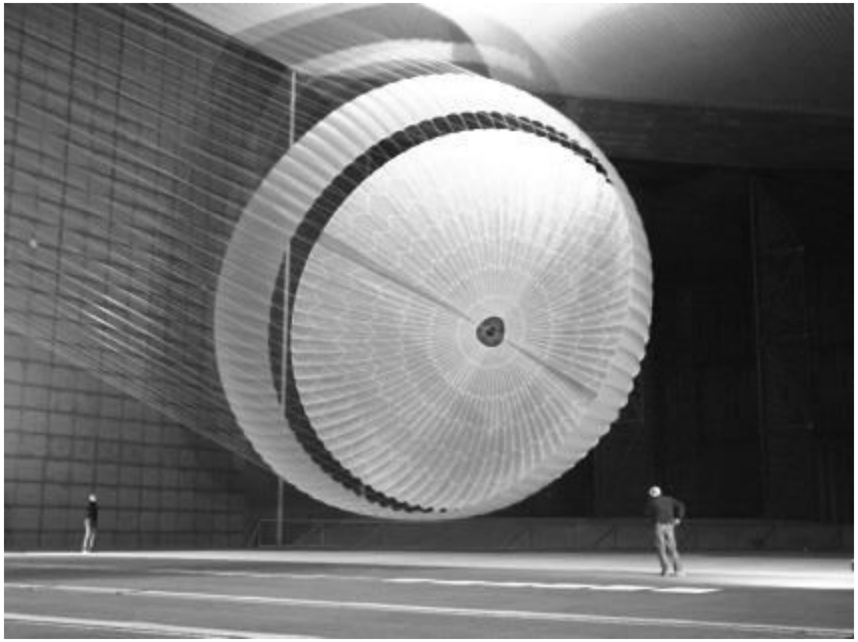




$$q'' = h(T_s - T_\infty)$$

Necessity of dimensional analysis

- Convection heat transfer depends on characteristics of fluid flow
- Most practical fluid flow problems are too complex, both geometrically and physically, to be solved analytically. Experimental testing or CFD analysis is resorted to approximate the solution to analytical problems
- The solution for such complex fluid problems is typically reported as experimental or numerical data points and smoothed curves. *Such data has more generality if expressed in compact form and hence is the motivation for dimensional analysis*
- *Dimensional analysis is a mainstay of fluid mechanics and is also widely used in all engineering fields*



Large parachute being tested at NASA facility for MARS mission

Introduction of dimensional analysis

- *Dimensional analysis is a method for reducing the number and complexity of experimental variables that affect a given physical phenomenon through compacting technique*
- If a fluid flow phenomenon depends on *n* dimensional variables, dimensional analysis will reduce the problem to only *k dimensionless* variables, where the reduction *n* − *k* = 1, 2, 3, or 4, depending on the problem complexity
- *Generally n − k equals number of basic or fundamental dimensions. In fluid mechanics problems, four basic dimensions are Mass (M), Length (L), Time (T) and temperature Θ*
- *Suppose the force F on a particular body shape immersed in a stream of fluid depended only on the body length L, stream velocity V, fluid density ρ , and fluid viscosity μ:*

$$F = f(L, V, \rho, \mu)$$

- *If the geometry and flow conditions are complicated and the analytical solutions are not feasible the function f(L, V,ρ ,μ) must be determined experimentally or numerically*

$$\boxed{\begin{gathered}\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \\ C_F = g(\mathrm{Re})\end{gathered}}$$

Contd------

Introduction of dimensional analysis

- *Some fluid forces have a very weak or negligible Reynolds number dependence in wide region. The force coefficient in such cases may depend, in high-speed gas flow, on the Mach number, Ma = V/a, where a is the speed of sound*

- In free-surface flows, such as ship drag, C_F may depend upon *Froude number*, Fr = $V^2/(gL)$, where g is the acceleration of gravity

- In turbulent flow, force may depend upon the *roughness ratio*, ε/L, where ε is the roughness height of the surface

- The function g is different mathematically from the original function f, but it contains all the same information. The experimental cost is minimized

- *Dimensional analysis will often give a great deal of insight into the form of the physical relationship of the variables of the problem*

- *Dimensional analysis provides scaling laws that can convert data from a cheap, small model to design information for an expensive, large prototype*

$$\text{If} \quad \mathrm{Re}_m = \mathrm{Re}_p \quad \text{then} \quad C_{Fm} = C_{Fp}$$

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

Principle of Dimensional Homogeneity

If an equation truly expresses a proper relationship between variables in a physical process, it will be *dimensionally homogeneous;* that is, each of its additive terms will have the same dimensions.

Consider the relation that expresses the displacement of a falling body

$$S = S_0 + V_0 t + \tfrac{1}{2} g t^2$$

Each term in this equation is a displacement, or length, and has dimensions {L}

Consider Bernoulli's equation for incompressible flow:

$$\frac{p}{\rho} + \frac{1}{2} V^2 + g z = \text{const}$$

Each term, including the constant, has dimensions of velocity squared, or $\{L^2 T^{-2}\}$

The motive behind dimensional analysis is that any dimensionally homogeneous equation can be written in an entirely equivalent nondimensional form that is more compact.

Variables and constants

Consider the following equations:

$$S = S_0 + V_0 t + \tfrac{1}{2} g t^2$$

$$\frac{p}{\rho} + \frac{1}{2} V^2 + g z = \text{const}$$

- *Dimensional variables are the quantities that actually vary during a given case and would be plotted against each other to show the data. Ex: S, t and p, V, z*
- *Dimensional constants may vary from case to case but are held constant during a given run. Ex: S_0, V_0, g and ρ, g , C. They all have dimensions and conceivably could be nondimensionalized, but they are normally used to help nondimensionalize the variables in the problem*
- *Pure constants have no dimensions and never did. They arise from mathematical manipulation. Angles and revolutions are dimensionless. Similarly counting numbers are also dimensionless*

$$\int S\,dt = S_0 t + \tfrac{1}{2} V_0 t^2 + \tfrac{1}{6} g t^3$$

$$\frac{dS}{dt} = V_0 + g t$$

- *Integration and differentiation may change dimensions not homogeneity of Eq.*

Choice of variables and scaling parameters in dimensional analysis

- Consider the equation of displacement of falling body again:

$$S = S_0 + V_0 t + \frac{1}{2} g t^2$$

- *The above equation can be divided into basic variables and parameters:*

Here S and t are variables , S_0, V_0 and g are parameters

- *To nondimensionalize the displacement equation of falling body , first it needs to be checked how many dimensions are there in variables and parameters:*

$$\{S\} = \{S_0\} = \{L\} \qquad \{t\} = \{T\} \qquad \{V_0\} = \{LT^{-1}\} \qquad \{g\} = \{LT^{-2}\}$$

- Therefore select two parameters to be scaling parameters or repeating variables

For the falling-body problem, selection of any two of the three parameters to be scaling parameters leads to following three options

Choice of variables and scaling parameters in dimensional analysis

Option 1: Scaling parameters S_0 and V_0: the effect of gravity g

Let S* and t* be dimensionless parameters:

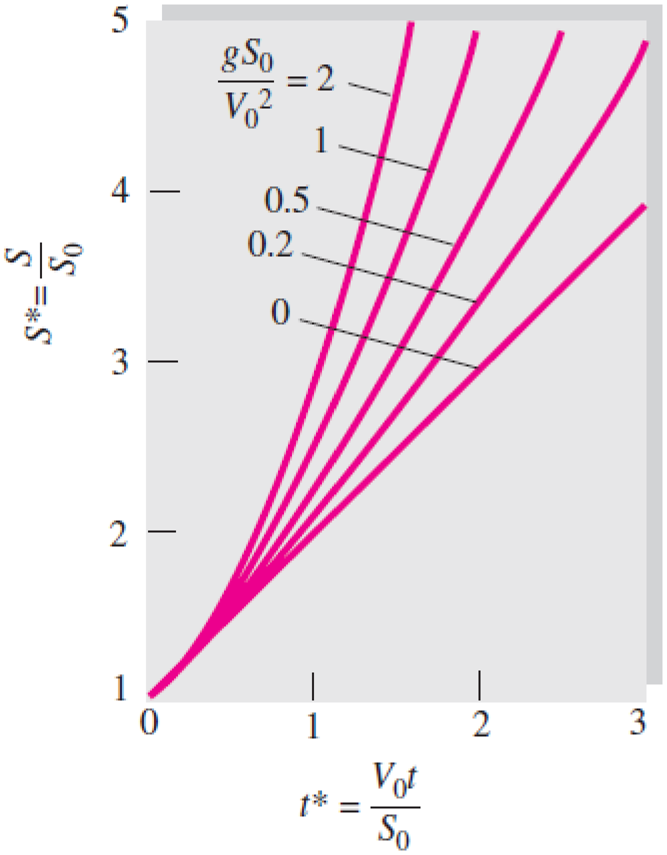
$$S^* = \frac{S}{S_0} \qquad t^* = \frac{V_0 t}{S_0}$$

Then free falling problem equation can be written as:

$$S^* = 1 + t^* + \frac{1}{2}\alpha t^{*2} \qquad \alpha = \frac{gS_0}{V_0^2}$$

There is a single dimensionless parameter α, which shows here the effect of gravity

Gravity increases the parabolic rate of fall for $t^* > 0$, but not the initial slope at $t^* = 0$.



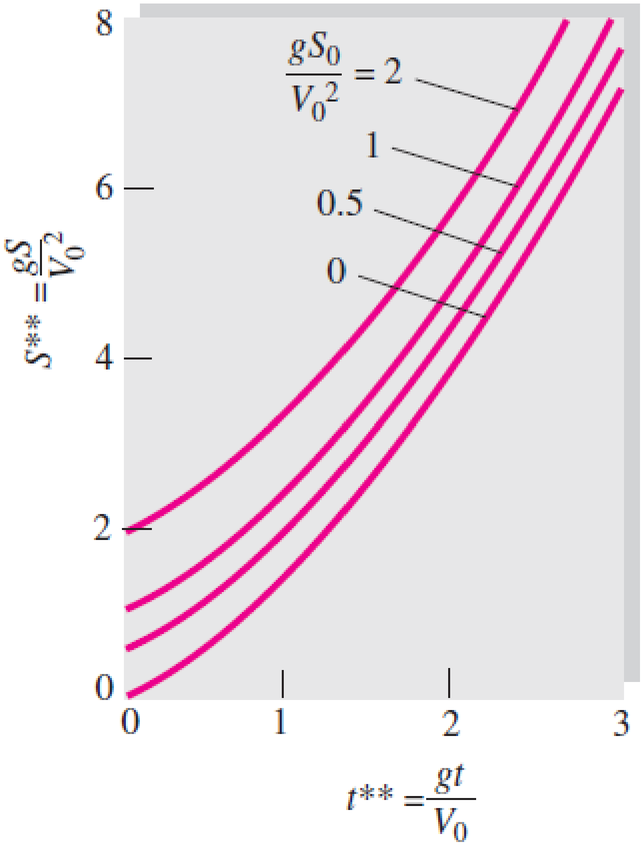
Choice of variables and scaling parameters in dimensional analysis

Option 2: Scaling parameters V_0 and g: the effect of initial displacement S_0.

$$S^{**} = \frac{Sg}{V_0^2} \qquad t^{**} = t\frac{g}{V_0}$$

$$S^{**} = \alpha + t^{**} + \frac{1}{2}t^{**2} \qquad \alpha = \frac{gS_0}{V_0^2}$$

The same single parameter α again appears and here shows the effect of initial displacement, which merely moves the curves upward without changing their shape



Choice of variables and scaling parameters in dimensional analysis

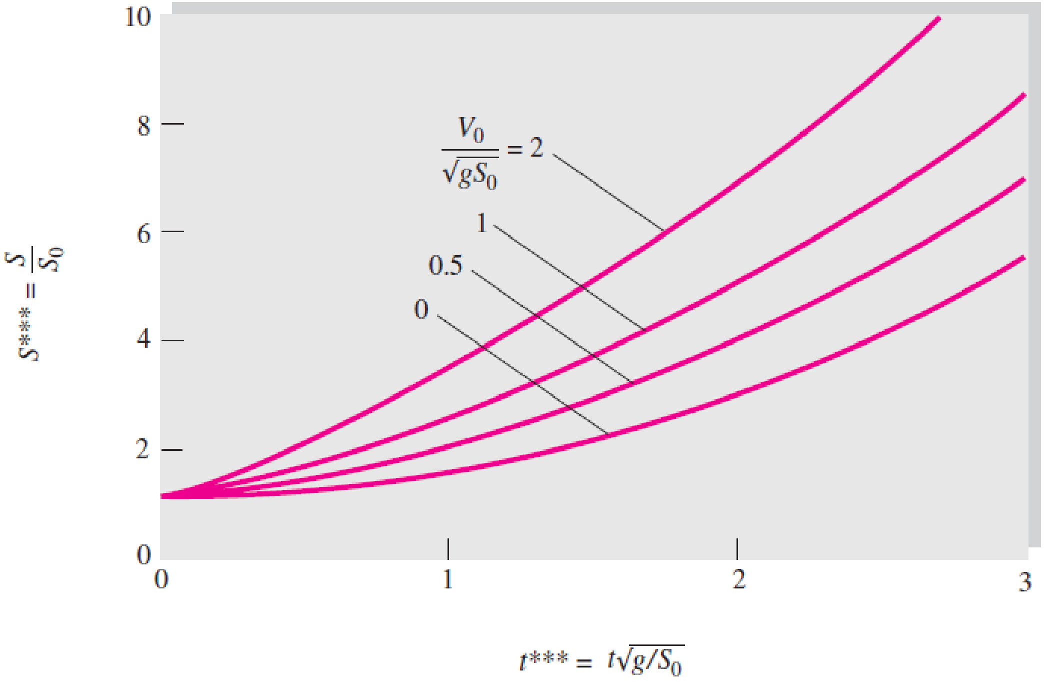
Option 3: Scaling parameters S_0 and g: the effect of initial speed V_0

The dimensionless parameters defined here are:

$$S^{***} = \frac{S}{S_0} \qquad t^{***} = t\left(\frac{g}{S_0}\right)^{1/2}$$

$$S^{***} = 1 + \beta t^{***} + \frac{1}{2} t^{***2} \qquad \beta = \frac{1}{\sqrt{\alpha}} = \frac{V_0}{\sqrt{gS_0}}$$

in all three options, the same parameter α appears but has a different meaning: dimensionless gravity, initial displacement, and initial velocity



Note the number of dimensions have been reduced from 5 to 2

$$S' = \text{fcn}(t', \alpha) \qquad \alpha = \frac{gS_0}{V_0^2}$$

Guidelines on selection of scaling variables

The following are some guidelines for selecting scaling variables:

- They must *not* form a dimensionless group among themselves, but adding one more variable *will* form a dimensionless quantity. For example:

$$\rho^a V^b L^c = (ML^{-3})^a (L/T)^b (L)^c = M^0 L^0 T^0 \quad \text{only if} \quad a = 0,\ b = 0,\ c = 0$$

- Do not select output variables for your scaling parameters

- If convenient, select *popular,* not obscure, scaling variables because they will appear in all dimensionless groups. For example select density, not surface tension, select body length, not surface roughness, select stream velocity, not speed of sound

The foundation of the dimensional analysis method rests on two assumptions: (1) The proposed physical relation is dimensionally homogeneous, and (2) all the relevant variables have been included in the proposed relation

Buckingham Pi theorem

It was introduced by Buckingham in 1914:

- *The name pi comes from the mathematical notation , meaning a product of variables. The dimensionless groups found from the theorem are product groups denoted by π_1, π_2, π_3, ---.*
- *The method allows the pi groups to be found in sequential order without resorting to free exponents*

The first part of the pi theorem explains what reduction in variables to expect:

If a physical process satisfies the PDH and involves n dimensional variables, it can be reduced to a relation between only k dimensionless variables or π's. The reduction $j = n - k$ equals the maximum number of variables that do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Contd------

Buckingham Pi theorem

The second part of the theorem shows how to find the pi groups one at a time:

Find the reduction j, then select j scaling variables that do not form a pi among themselves. Each desired pi group will be a power product of these j variables plus one additional variable, which is assigned any convenient nonzero exponent. Each pi group thus found is independent

- *Suppose a process involves five variables:*

$$v_1 = f(v_2, v_3, v_4, v_5)$$

- *Suppose there are three dimensions {MLT} and hence j = 3. Then k = 5 − 3 = 2*
- *Pick out three convenient variables that do not form a pi, and suppose these turn out to be v_2, v_3, and v_4. Then the two pi groups are formed by power products of these three plus one additional variable, either v_1 or v_5*

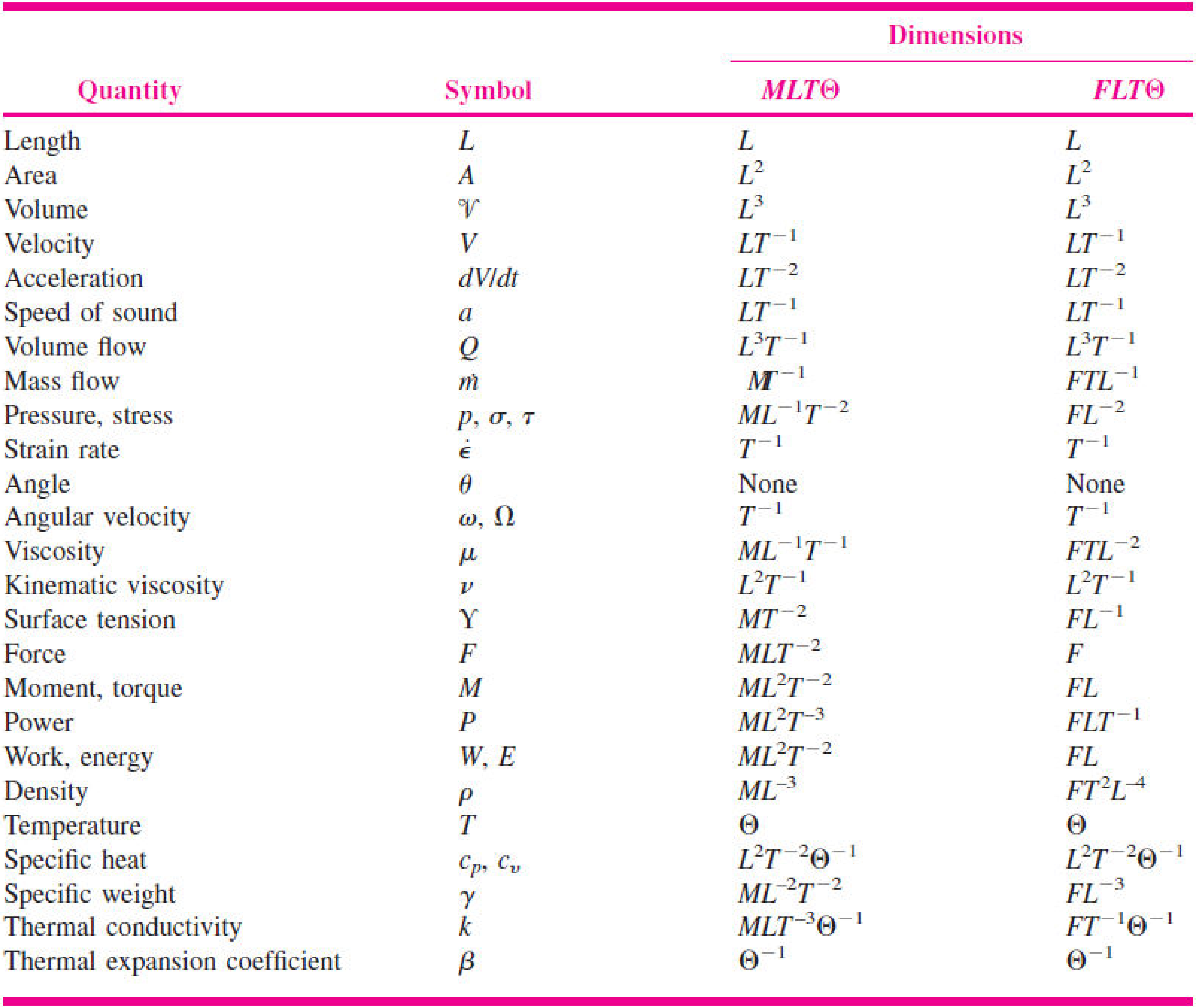
$$\Pi_1 = (v_2)^a (v_3)^b (v_4)^c v_1 = M^0 L^0 T^0 \qquad \Pi_2 = (v_2)^a (v_3)^b (v_4)^c v_5 = M^0 L^0 T^0$$

- *Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of a, b, and c for each pi. And they are independent because only π_1 contains v_1 and only π_2 contains v_5*

Summary of steps involved in finding π products

- *List and count the n variables involved in the problem. If any important variables are missing, dimensional analysis will fail*
- *List the dimensions of each variable according to {MLTΘ} or {FLTΘ}.*
- *Find j. Initially guess j equal to the number of different dimensions present, and look for j variables that do not form a pi product. If no luck, reduce j by 1 and look again. With practice, you will find j rapidly*
- *Select j scaling parameters that do not form a pi product. Make sure they have some generality if possible*
- *Add one additional variable to your j repeating variables, and form a power product. Algebraically find the exponents that make the product dimensionless. Try to arrange for your output or dependent variables. Do this sequentially adding one new variable at a time, find all n−j = k pi products*
- *Write the final dimensionless function, and check the terms to make sure all pi groups are dimensionless*

Dimensions of Fluid Mechanics problems



		Dimensions	
Quantity	**Symbol**	***MLTΘ***	***FLTΘ***
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ, τ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω, Ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Thermal expansion coefficient	β	Θ^{-1}	Θ^{-1}

Dimensional analysis - an example

- *Suppose the force F on a particular body shape immersed in a stream of fluid depended only on the body length L, stream velocity V, fluid density ρ , and fluid viscosity μ:* $F = f(L, V, \rho, \mu)$

- *The final numerical relation between variables is given by :*

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right)$$

$$C_F = g(\mathrm{Re})$$

Step 1

Write the function and count variables:

$$F = f(L, U, \rho, \mu) \quad \text{there are five variables } (n = 5)$$

Step 2

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

Contd------

Dimensional analysis - an example

Step 3

Find j. No variable contains the dimension Θ, and so j is less than or equal to 3 (MLT). We inspect the list and see that L, U, and ρ cannot form a pi group because only ρ contains mass and only U contains time. Therefore j does equal 3, and $n - j = 5 - 3 = 2 = k$. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

Step 4

Select repeating j variables. The group L, U, ρ we found in step 3

Step 5

Combine L, U, ρ with one additional variable, in sequence, to find the two pi products.

First add force to find Π_1. You may select *any* exponent on this additional term as you please, to place it in the numerator or denominator to any power. Since F is the output, or dependent, variable, we select it to appear to the first power in the numerator:

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

Length: $a + b - 3c + 1 = 0$

Mass: $c + 1 = 0$

Time: $-b \qquad -2 = 0$

Solving algebraically: $a = -2 \qquad b = -2 \qquad c = -1$

Dimensional analysis - an example

$$\Pi_1 = L^{-2}U^{-2}\rho^{-1}F = \frac{F}{\rho U^2 L^2} = C_F$$

Finally, add viscosity to L, U, and to find π_2. Select any power you like for viscosity. By hindsight and custom, we select the power −1 to place it in the denominator

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

Length: $a + b - 3c + 1 = 0$

Mass: $c - 1 = 0$

Time: $-b \qquad + 1 = 0$

Solving algebraically: $a = b = c = 1$

$$\Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \frac{\rho U L}{\mu} = \mathrm{Re}$$

Step 6 $$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right)$$

Dimensional analysis - Second example

The power input P to a centrifugal pump is a function of the volume flow Q, impeller diameter D, rotational rate Ω, and the density ρ, and viscosity of the fluid μ

$$P = f(Q, D, \Omega, \rho, \mu)$$

Hint: Consider Ω, ρ , and *D as repeating variables*

Solution:

Step 1: Count the variables. There are six (don't forget the one on the left, P)

Step 2:

P	Q	D	Ω	ρ	μ
$\{FLT^{-1}\}$	$\{L^3T^{-1}\}$	$\{L\}$	$\{T^{-1}\}$	$\{FT^2L^{-4}\}$	$\{FTL^{-2}\}$

Step 3: (Ω, ρ, D) as repeating variables

Check that these three do *not* form a pi group

$$\Omega^a \rho^b D^c = (T^{-1})^a (FT^2L^{-4})^b (L)^c = F^0L^0T^0 \quad \text{only if} \quad a = 0,\ b = 0,\ c = 0$$

Contd------

Dimensional analysis - Second example

Step 4a: Combine (Ω, ρ, D) with power P to find the first pi group:

$$\Pi_1 = \Omega^a \rho^b D^c P = (T^{-1})^a (FT^2L^{-4})^b (L)^c (FLT^{-1}) = F^0 L^0 T^0$$

By equating exponents:

Force: $b + 1 = 0$

Length: $-4b + c + 1 = 0$

Time: $-a + 2b - 1 = 0$

Solving algebraically: $a = -3$, $b = -1$, and $c = -5$

This first pi group, the output dimensionless variable, is called the *power coefficient* of a pump, C_P

$$\Pi_1 = \Omega^{-3}\rho^{-1}D^{-5}P = \frac{P}{\rho\Omega^3 D^5} = C_P$$

Contd------

Dimensional analysis - Second example

Step 4b: Combine (Ω, ρ, D) with flow rate Q to find the second pi group:

$$\Pi_2 = \Omega^a \rho^b D^c Q = (T^{-1})^a (FT^2L^{-4})^b (L)^c (L^3T^{-1}) = F^0L^0T^0$$

Solving algebraically: $a = -1$, $b = 0$, and $c = -3$.

The second pi group is called flow coefficient of the pump, C_Q

$$\Pi_2 = \Omega^{-1}\rho^0 D^{-3} Q = \frac{Q}{\Omega D^3} = C_Q$$

Step 4c: Combine (Ω, ρ, D) with viscosity μ to find the third and last pi group:

$$\Pi_3 = \Omega^a \rho^b D^c \mu = (T^{-1})^a (FT^2L^{-4})^b (L)^c (FTL^{-2}) = F^0L^0T^0$$

Solving algebraically: $a = -1$, $b = -1$, and $c = -2$; or $\Pi_3 = \mu/(\rho\Omega D^2)$

The original relation between six variables is reduced to three dimensionless groups:

$$\frac{P}{\rho\Omega^3 D^5} = f\left(\frac{Q}{\Omega D^3}, \frac{\mu}{\rho\Omega D^2}\right)$$

Step by step method of Ipsen for estimating non dimensional quantities

Pi theorem has some inherent drawbacks:

- All π groups contain the same repeating variables and might lack effectiveness
- Involves laborious checking of repeating variables that donot form π groups

Ipsen suggests an alternative method to obtain π groups without any checking:

Consider the same classical drag function of body submerged in a fluid:

$$\begin{array}{cccccc} F & = \text{fcn}(& L, & V, & \rho, & \mu) \\ \{MLT^{-2}\} & & \{L\} & \{LT^{-1}\} & \{ML^{-3}\} & \{ML^{-1}T^{-1}\} \end{array}$$

There are three dimensions, $\{MLT\}$. Eliminate them successively by division or multiplication by a variable. Start with mass $\{M\}$. Pick a variable that contains mass and divide it into all the other variables with mass dimensions

$$\begin{array}{cccccc} \dfrac{F}{\rho} & = \text{fcn}\Bigg(& L, & V, & \cancel{\rho,} & \dfrac{\mu}{\rho}\Bigg) \\ \{L^{4}T^{-2}\} & & \{L\} & \{LT^{-1}\} & \{ML^{-3}\} & \{L^{2}T^{-1}\} \end{array}$$

Step by step method of Ipsen for estimating non dimensional quantities

- *Discard ρ*, and now there are only four variables. Next, eliminate time {*T*}

$$\underset{\{L^2\}}{\frac{F}{\rho V^2}} = \text{fcn}\left(\underset{\{L\}}{L,} \quad \underset{\{LT^{-1}\}}{\cancel{V},} \quad \underset{\{L\}}{\frac{\mu}{\rho V}}\right)$$

Now we see that V is no longer relevant since only V contains time {T}. Finally, eliminate {L} through division by, say, appropriate powers of L itself

$$\underset{\{1\}}{\frac{F}{\rho V^2 L^2}} = \text{fcn}\left(\underset{\{L\}}{\cancel{L},} \quad \underset{\{1\}}{\frac{\mu}{\rho V L}}\right)$$

HEAT TRANSFER - 80329

MODULE III

FUNDAMENTALS OF CONVECTIVE HEAT TRANSFER

Concepts of continuity, momentum and energy equations

Part 3

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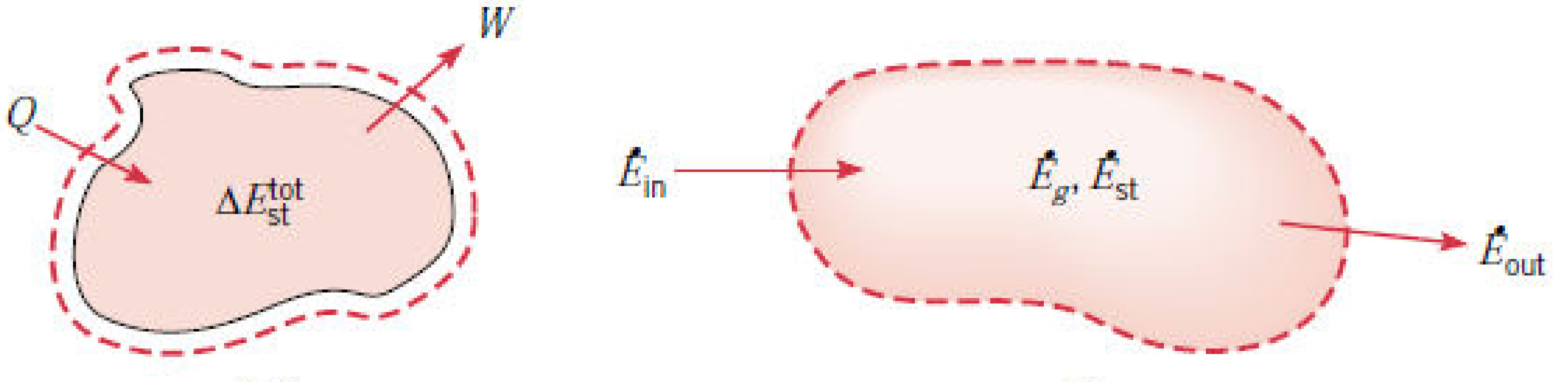
Heat Equation- convection

- Previously in developing heat rate equation for conduction we applied conservation of energy for a differential control volume. For a prescribed geometry and boundary conditions the equations must be solved to determine the temperature distribution

- In case of moving fluid the effects of fluid motion on energy transfer across the surfaces of control volume must be considered along with those of conduction
- For convective heat transfer the solution of differential equation that provides temperature distribution requires knowledge of velocity field
- Here we consider flow of viscous fluid in which there is concurrent heat and mass transfer and by using Newtons laws of motion and conservation of mass and energy to predict velocity, temperature and mass concentration fields in a fluid



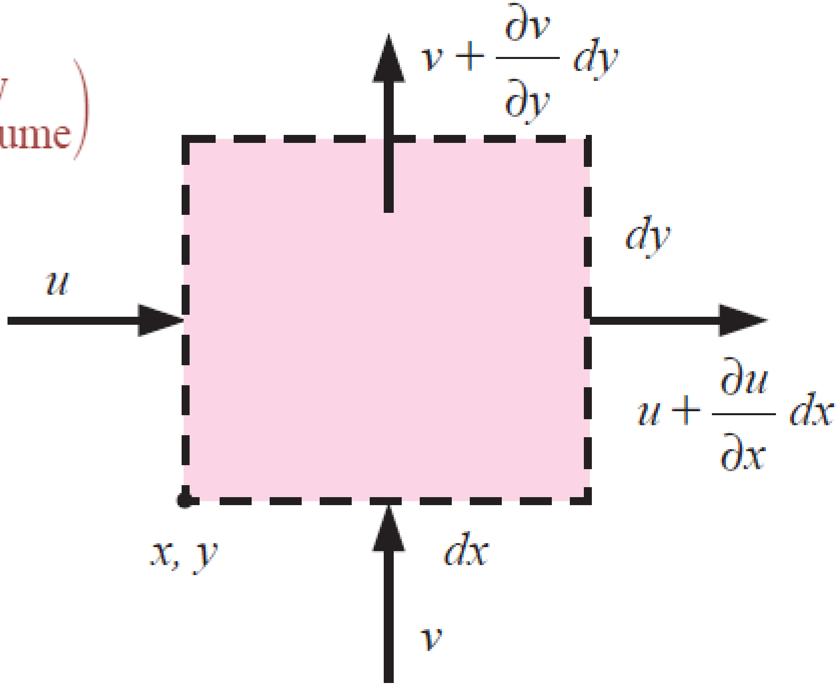
(a) (b)

Conservation of energy for a) closed system over a time interval b) for control volume at an instant

Conservation of mass equation

- Here we restrict our attention to steady two-dimensional flow in x and y directions of a Cartesian coordinate system. Unit depth is assumed in *z* direction therefore the differential control volume of the element is (dx. dy. 1)
- The conservation of mass principle is simply a statement that mass cannot be created or destroyed, and all the mass must be accounted for during an analysis
- In steady flow, the amount of mass within the control volume remains constant, and thus the conservation of mass can be expressed as:

$$\begin{pmatrix}\text{Rate of mass flow}\\ \text{into the control volume}\end{pmatrix} = \begin{pmatrix}\text{Rate of mass flow}\\ \text{out of the control volume}\end{pmatrix}$$



Noting that mass flow rate is equal to the product of density, mean velocity, and cross-sectional area normal to flow, the rate at which fluid enters the control volume from the left surface is $\rho u(dy \,.\, 1)$

Conservation of mass equation

- The rate at which the fluid leaves the control volume from the right surface can be expressed as:

$$\rho\left(u + \frac{\partial u}{\partial x}dx\right)(dy \cdot 1)$$

- The rate at which fluid enters and leaves control volume in y direction is:

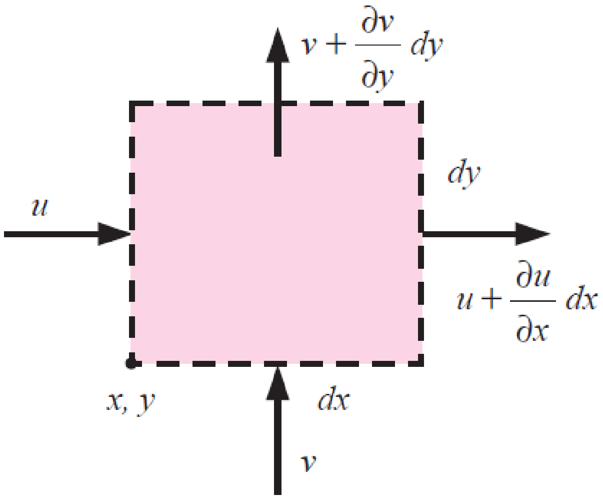
$\rho v(dx \,.\, 1)$ and $\rho\left(v + \frac{\partial v}{\partial y}dy\right)(dx \cdot 1)$

- On application of law of conservation of mass:

$$\rho u(dy \cdot 1) + \rho v(dx \cdot 1) = \rho\left(u + \frac{\partial u}{\partial x}dx\right)(dy \cdot 1) + \rho\left(v + \frac{\partial v}{\partial y}dy\right)(dx \cdot 1)$$

Simplifying and dividing by *dx . dy .* 1 gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



This is the *conservation of mass* relation, also known as the **continuity equation,** or **mass balance** for steady two dimensional flow of a fluid with constant density

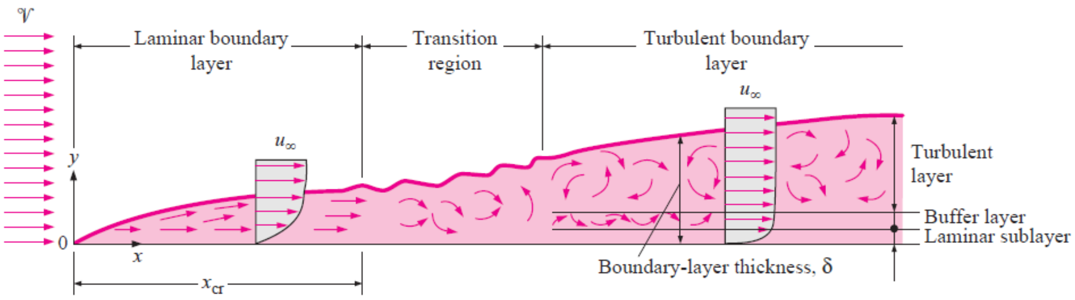
Conservation of momentum equation

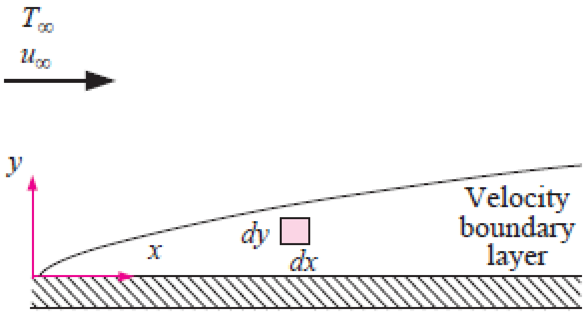
- ❖ The differential forms of the equations of motion in the velocity boundary layer are obtained by applying Newton's second law of motion to a differential control volume element in the boundary layer

- ❖ *According to Newton's second law of motion, the net force acting on the control volume is equal to the mass times the acceleration of the fluid element within the control volume, which is also equal to the net rate of momentum outflow from the control volume*

- ❖ The forces acting on the control volume are body forces that act through the entire body of control volume and are proportional volume of the body. While surface forces act on a control surface and are proportional to surface area

$$(\text{Mass})\left(\begin{array}{c}\text{Acceleration}\\ \text{in a specified direction}\end{array}\right) = \left(\begin{array}{c}\text{Net force (body and surface)}\\ \text{acting in that direction}\end{array}\right) \Rightarrow \delta m \cdot a_x = F_{\text{surface},\,x} + F_{\text{body},\,x}$$





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Conservation of momentum equation

- *Mass of the fluid element within the control volume is:*

$$\delta m = \rho(dx \cdot dy \cdot 1)$$

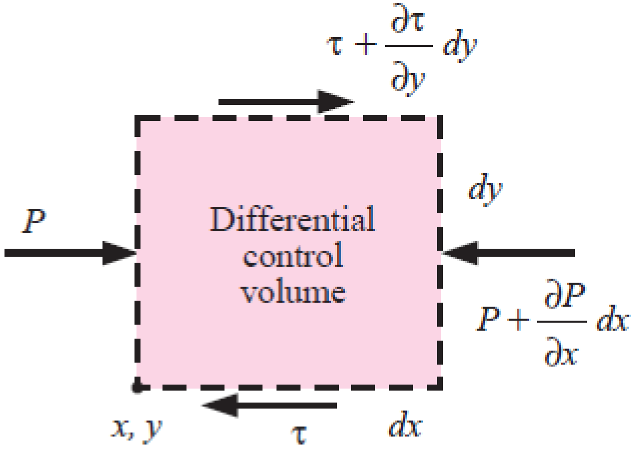
- *Since flow is two dimensional and u = u (x, y), total differential of u is :*

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

- *Acceleration of fluid in x direction is :*

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

The forces acting on a surface are due to pressure and viscous effects. In two-dimensional flow, the viscous stress at any point on an imaginary surface within the fluid can be resolved into two perpendicular components: one normal to the surface called normal stress (which should not be confused with pressure) and another along the surface called shear stress



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Conservation of momentum equation

- *Neglecting the normal stresses for simplicity, the surface forces acting on the control volume in the x-direction will be as shown in Fig. below:*

$$F_{\text{surface},\,x} = \left(\frac{\partial \tau}{\partial y}\,dy\right)(dx\cdot 1) - \left(\frac{\partial P}{\partial x}\,dx\right)(dy\cdot 1) = \left(\frac{\partial \tau}{\partial y} - \frac{\partial P}{\partial x}\right)(dx\cdot dy\cdot 1)$$

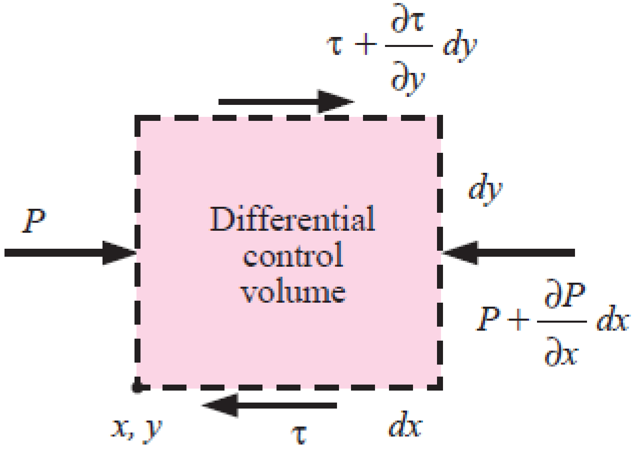
$$= \left(\mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}\right)(dx\cdot dy\cdot 1)$$

By substituting $\tau = \mu\,\frac{du}{dy}$

$$(\text{Mass})\begin{pmatrix}\text{Acceleration}\\ \text{in a specified direction}\end{pmatrix} = \begin{pmatrix}\text{Net force (body and surface)}\\ \text{acting in that direction}\end{pmatrix}$$

$$\rho\left(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right) = \mu\,\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}$$

This is the relation for the ***conservation of momentum*** *in the x-direction, and is known as the* ***x-momentum equation.*** *If there is a body force acting in the x-direction, it can be added to the right side of the equation provided that it is expressed per unit volume of the fluid*



Conservation of Energy equation

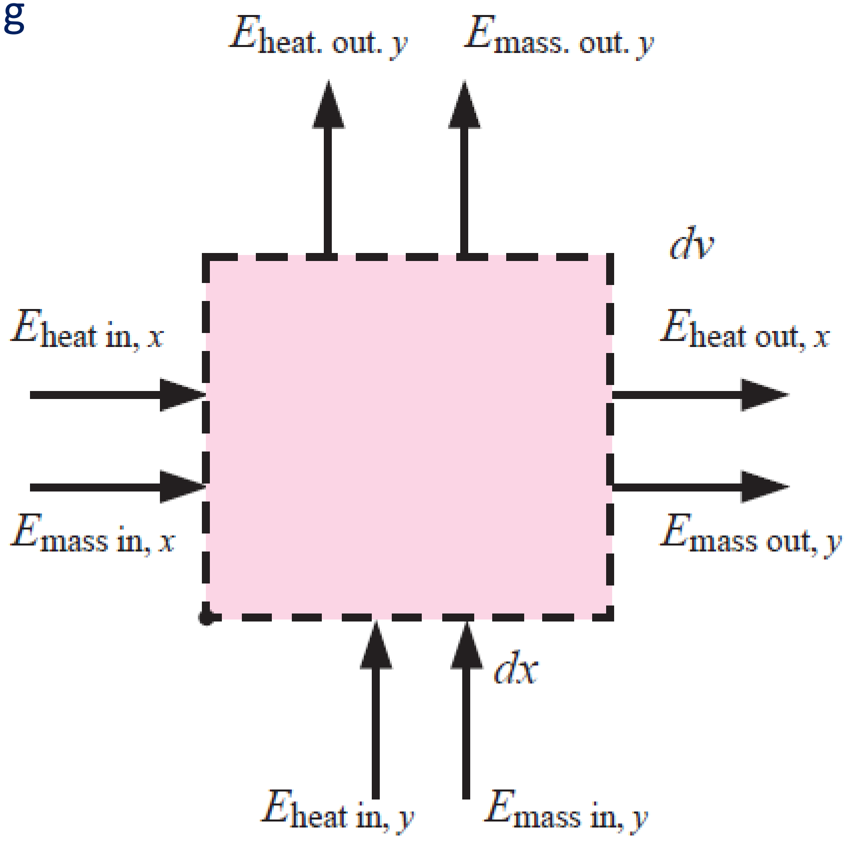
- The energy balance for any system undergoing any process is expressed as:

$$E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{system}}$$

- For a steady flow process the general energy equation reduces to:

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0$$

- Since energy can be transferred by heat, work, and mass only, the energy balance for a steady-flow control volume can be written explicitly as:

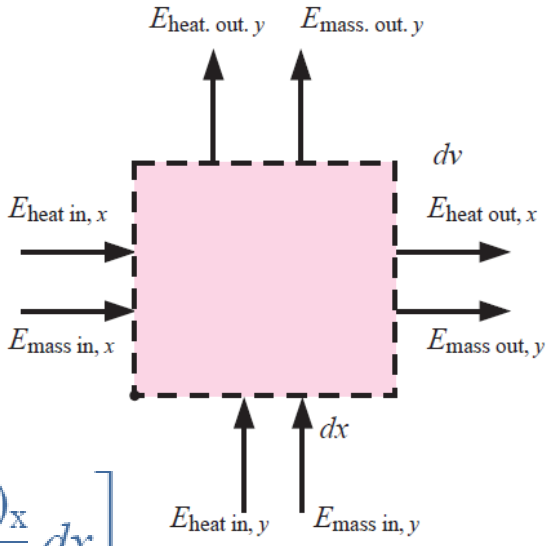


$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by heat}} + (\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by work}} + (\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by mass}} = 0$$

The total energy of a flowing fluid stream per unit mass is e_{stream} *=h +ke +pe where h is the enthalpy (which is the sum of internal energy and flow energy),pe = gz is the potential energy. Compared to enthalpy K.E and P.E is small and neglected*

Contd---

Conservation of Energy equation



Energy is a scalar quantity, and thus energy interactions in all directions can be combined in one equation

- Mass flow rate of the fluid entering the control volume from the left is ρ*u*(*dy* . 1)
- Rate of energy transfer to the control volume by mass in the *x*-direction is:

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by mass, } x} = (\dot{m}e_{\text{stream}})_x - \left[(\dot{m}e_{\text{stream}})_x + \frac{\partial(\dot{m}e_{\text{stream}})_x}{\partial x}\,dx\right]$$

$$= -\frac{\partial[\rho u(dy\cdot 1)C_pT]}{\partial x}\,dx = -\rho C_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dx\,dy$$

Repeating the above exercise in y-direction and adding both x an y terms :

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by mass}} = -\rho C_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dx\,dy - \rho C_p\left(v\frac{\partial T}{\partial y} + T\frac{\partial v}{\partial y}\right)dx\,dy$$

$$= -\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)dx\,dy$$

Contd---

Conservation of Energy equation

The net rate of heat conduction to the volume element in the *x*-direction is:

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by heat, }x} = \dot{Q}_x - \left(\dot{Q}_x + \frac{\partial Q_x}{\partial x}dx\right)$$

$$= -\frac{\partial}{\partial x}\left(-k(dy \cdot 1)\frac{\partial T}{\partial x}\right)dx = k\frac{\partial^2 T}{\partial x^2}dx\,dy$$

Repeating this for the y-direction and adding the results, the net rate of energy transfer to the control volume by heat conduction becomes:

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by heat}} = k\frac{\partial^2 T}{\partial x^2}dx\,dy + k\frac{\partial^2 T}{\partial y^2}dx\,dy = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)dx\,dy$$

Ignore the work done by body forces and viscous forces. The work done by pressure force is flow work which is already included in energy transferred by mass stream the 2D energy equation is given by:

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

The net energy convected by the fluid out of the control volume is equal to the net energy transferred into the control volume by heat conduction

Conservation of Energy equation

- When the viscous shear stresses are not negligible, their effect is accounted for by expressing the energy equation as:

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi$$

- Viscous dissipation function ϕ play a dominant role in high-speed flows, especially when the viscosity of the fluid is high (like the flow of oil in journal bearings). This manifests itself as a significant rise in fluid temperature due to the conversion of the kinetic energy of the fluid to thermal energy.

- Viscous dissipation is also significant for high-speed flights of aircraft

- For the special case of a stationary fluid , u = v = 0, energy equation reduces to a 2D heat conduction equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

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MODULE III
FUNDAMENTALS OF CONVECTIVE HEAT TRANSFER

Boundary layer theory - Non dimensional correlation for Convective Heat transfer

Part 4

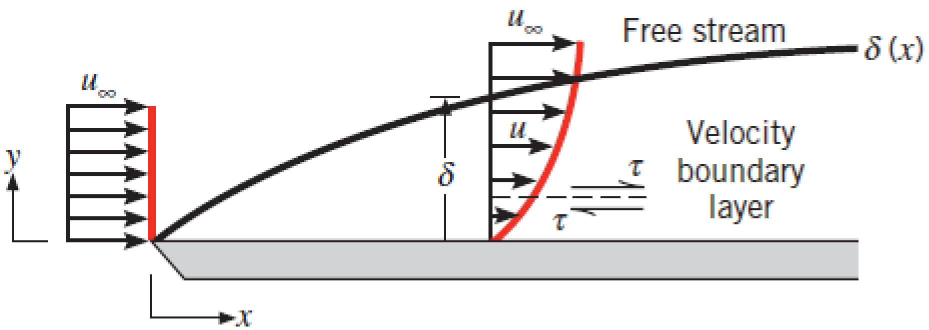
Dr. Satish Idury
Associate Professor

Velocity boundary layer

- When fluid particles make contact with the surface, their velocity is reduced significantly relative to the fluid velocity upstream of the plate, and for most situations it is valid to assume that the particle velocity is zero at the wall
- Fluid particles act to retard the motion of particles in the adjoining fluid layer, which act to retard the motion of particles in the next layer and so on until at y = δ from the surface the effects become negligible
- The quantity δ is termed the *boundary layer thickness*, and it is typically defined as the value of *y* for which $u = 0.99u_\infty$. The *boundary layer velocity profile* refers to the manner in which *u* varies with *y* through the boundary layer.
- With increasing distance from the leading edge, the effects of viscosity penetrate farther into the free stream and the boundary layer grows (δ increases with *x*)

$$C_f \equiv \frac{\tau_s}{\rho u_\infty^2/2}$$

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

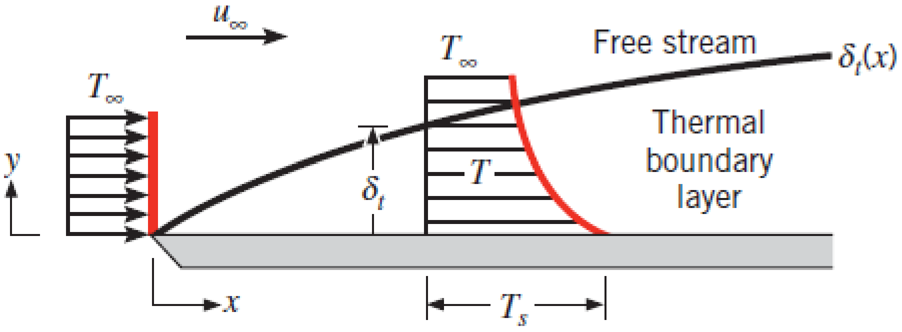


Velocity boundary layer development on a flat plate

Thermal boundary layer

- Consider a flow over an isothermal plate. A thermal boundary layer develops if the fluid free stream and surface temperature differ. At leading edge the temperature profile is uniform
- Fluid particles in contact with plate achieve thermal equilibrium at the plate’s surface temperature. Temperature gradients develop within the fluid layers due to exchange of energy with adjoining layers
- The region of the fluid in which these temperature gradients exist is the thermal boundary layer, and its thickness δ_t is typically defined as the value of *y* for which the ratio is: $[(T_s - T)/(T_s - T_\infty)] = 0.99$
- The relation between conditions in this boundary layer and the convection heat transfer coefficient can be demonstrated from the relations below:

$$q_s'' = -k_f \left.\frac{\partial T}{\partial y}\right|_{y=0} \qquad q_s'' = h(T_s - T_\infty) \qquad \Rightarrow \qquad h = \frac{-k_f \, \partial T/\partial y\big|_{y=0}}{T_s - T_\infty}$$



Thermal boundary layer development on an isothermal flat plate

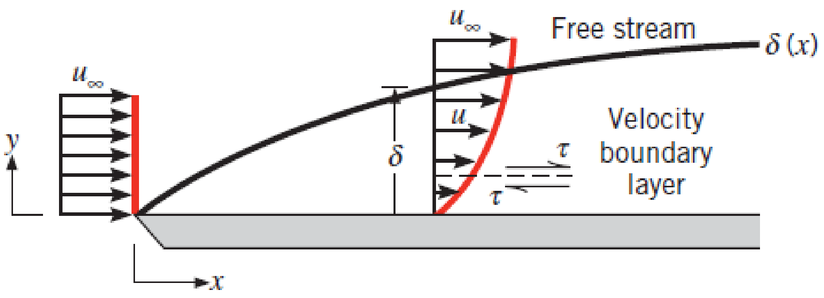
Significance of boundary layers

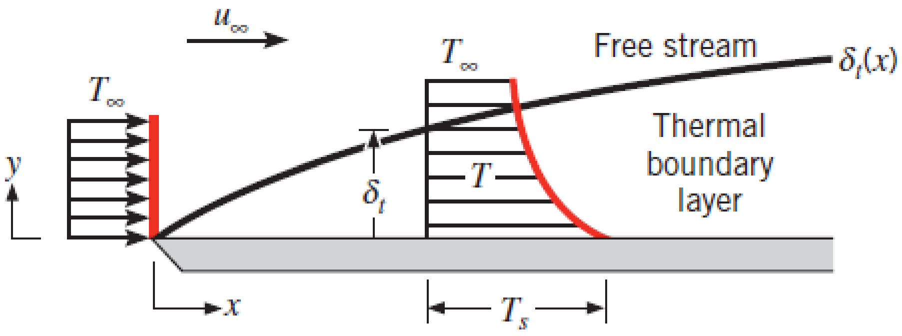
- For flow over any surface, there will always exist a velocity boundary layer and hence surface friction. The velocity boundary layer is of extent δ (*x*) and is characterized by the presence of velocity gradients and shear stresses

- A thermal boundary layer, and hence convection heat transfer, will always exist if the surface and free stream temperatures differ. The thermal boundary layer is of extent δ_t (x) and is characterized by temperature gradients and heat transfer

- The principal manifestations of velocity and thermal boundary layers are surface friction and convection heat transfer.

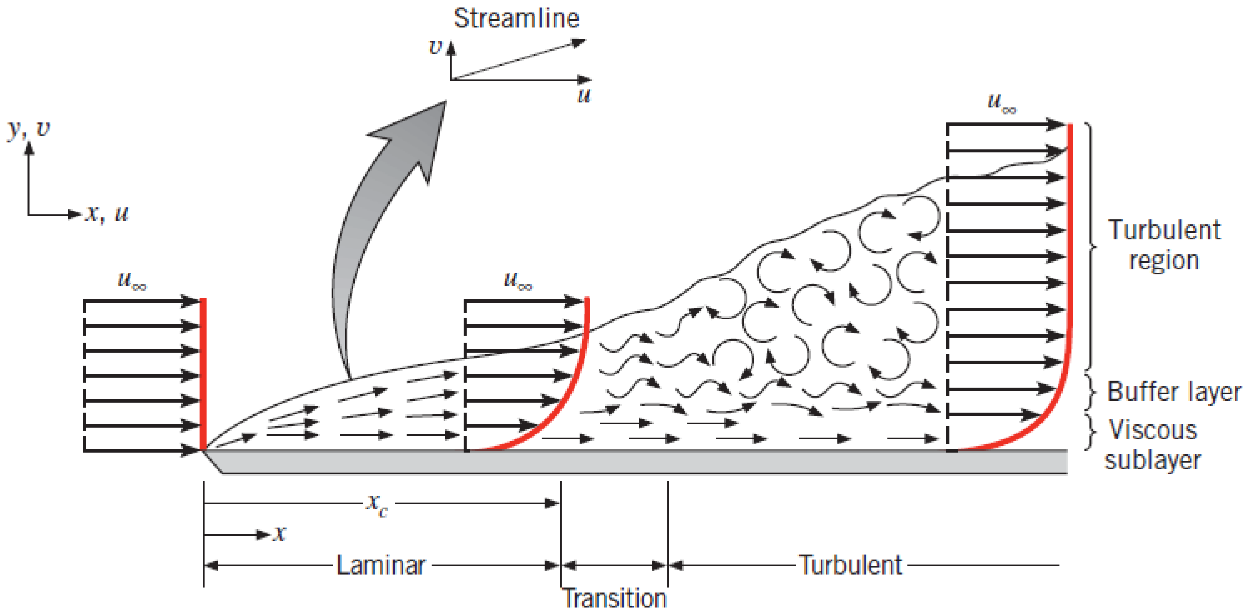
- The key boundary layer parameters are friction coefficient, C_f and convective heat transfer coefficient , h





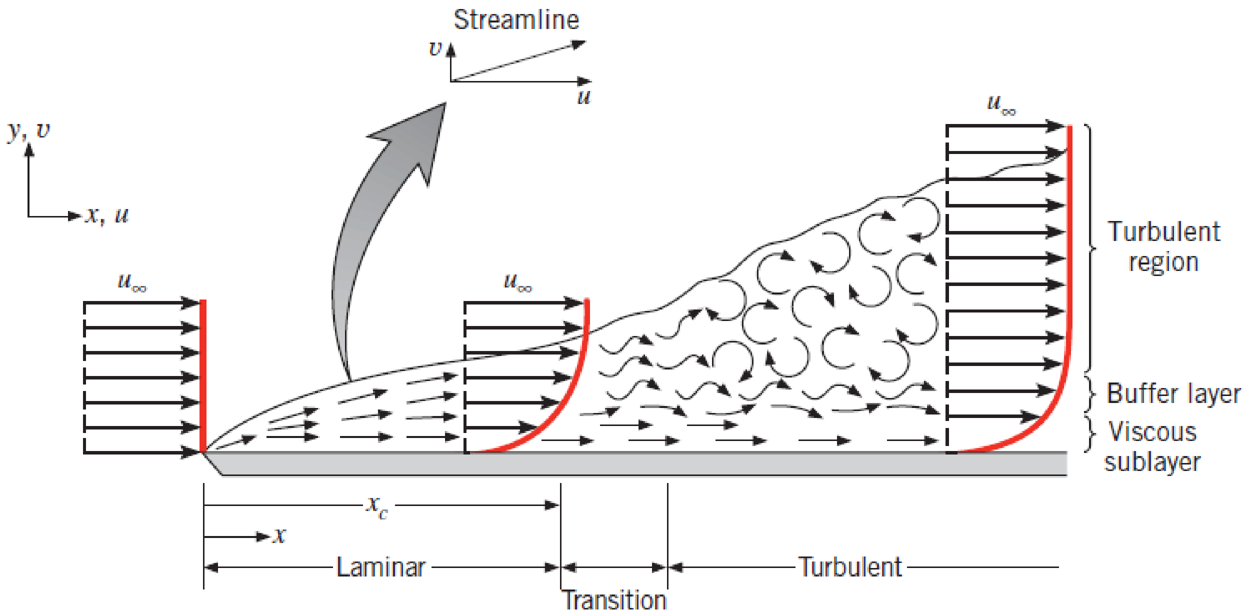
Laminar and turbulent velocity boundary layers

- Boundary layer *development* on a flat plate is illustrated in Fig. below In many cases, laminar and turbulent flow conditions both occur, with the laminar section preceding the turbulent section
- In the laminar boundary layer, the fluid flow is highly ordered and it is possible to identify streamlines along which fluid particles move. The boundary layer thickness grows and that velocity gradients at $y = 0$ decrease in the increasing x direction
- Local surface shear stress τ_s also decreases with increasing x. The highly ordered behavior continues until a *transition* zone is reached, across which a conversion from laminar to turbulent conditions occurs



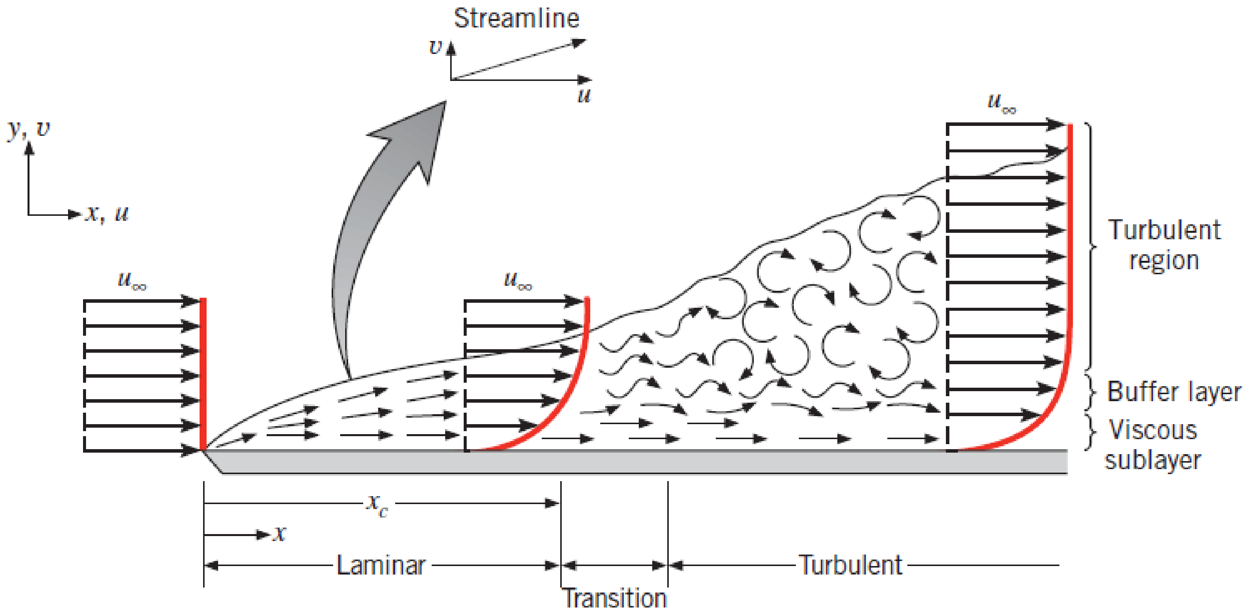
Laminar and turbulent velocity boundary layers

- Conditions within the transition zone change with time, with the flow sometimes exhibiting laminar behavior and sometimes exhibiting the characteristics of turbulent flow
- Flow in the fully turbulent boundary layer is, in general, highly irregular and is characterized by random, three-dimensional motion. Mixing within the boundary layer carries high-speed fluid toward the solid surface and transfers slower-moving fluid farther into the free stream
- Much of the mixing is promoted by stream wise vortices called *streaks* that are generated intermittently near the flat plate, where they rapidly grow and decay



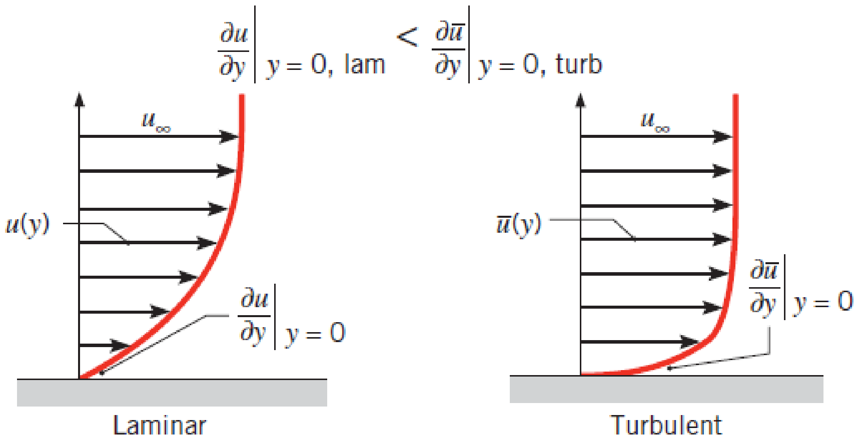
Laminar and turbulent velocity boundary layers

- As a result of the interactions that lead to chaotic flow conditions, velocity and pressure fluctuations occur at any point within the turbulent boundary layer
- Three different regions may be delineated within the turbulent boundary layer as a function of distance from the surface
- *In viscous sub-layer* transport is dominated by diffusion and the velocity profile is nearly linear
- *In buffer layer* diffusion and turbulent mixing are comparable, and there is a *turbulent zone* in which transport is dominated by turbulent mixing



Laminar and turbulent velocity boundary layers

- Fig. below shows that the turbulent velocity profile is relatively flat due to the mixing that occurs within the buffer layer and turbulent region, giving rise to large velocity gradients within the viscous sub layer
- τ_s is generally larger in the turbulent portion of the boundary layer of Fig. than in the laminar portion.
- The transition from laminar to turbulent flow is ultimately due to *triggering mechanisms*, such as the interaction of unsteady flow structures that develop naturally within the fluid or small disturbances that exist within many typical boundary layers
- The onset of turbulence depends on whether the triggering mechanisms are amplified or attenuated in the direction of fluid flow, which in turn depends on a dimensionless grouping of parameters called the *Reynolds number, Re_x*



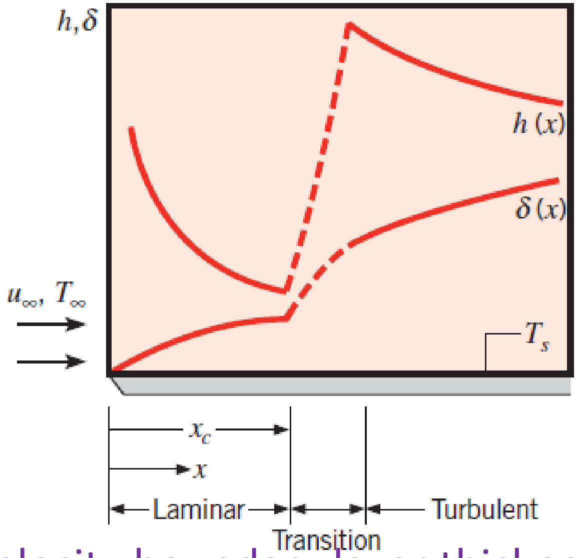
$$Re_x = \frac{\rho u_\infty x}{\mu}$$

Laminar and turbulent thermal boundary layers

- Since the velocity distribution determines the advective component of thermal energy transfer within the boundary layer, the nature of the flow also has a profound effect on convective heat transfer rates

- For laminar conditions, the thermal boundary layer grow in the stream wise (increasing *x*) direction. Hence the temperature gradient in the fluid decreases at y = 0 in the streamwise direction

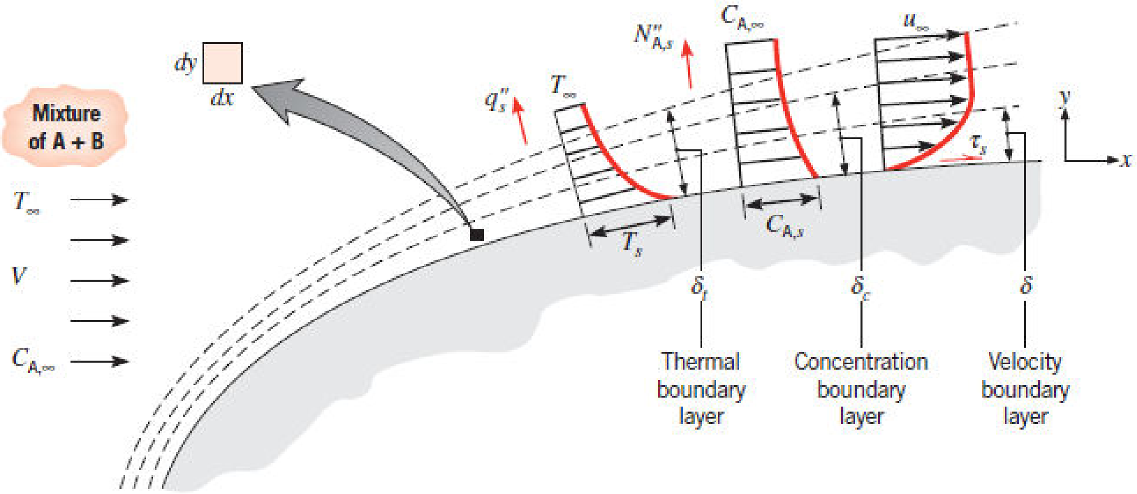
- Turbulent mixing promotes large temperature gradients adjacent to the solid surface and corresponding increase in heat transfer coefficient across the transition region
- Differences in thickness of velocity and thermal boundary layer tend to be smaller in turbulent flow



Variation of velocity boundary layer thickness δ and the local heat transfer coefficient *h* for flow over an isothermal flat plate

The Boundary layer equations

- The relevance of boundary layer to convection transport is depicted by considering the equations that govern boundary layer conditions
- The boundary layer equations will be used extensively to determine important dimensionless parameters associated with convection heat transfer
- Motion of a fluid in which there are coexisting velocity, temperature gradients must comply with several *fundamental laws of nature*. In particular, at each point in the fluid, *conservation of mass*, *energy*, as well as *Newton's second law of motion*, must be satisfied
- Equations representing these requirements are derived by applying the laws to a stationary differential control volume for the *steady*, *two-dimensional flow* of an *incompressible fluid* with *constant properties*



Development of the velocity, thermal, and concentration boundary layers for an arbitrary surface

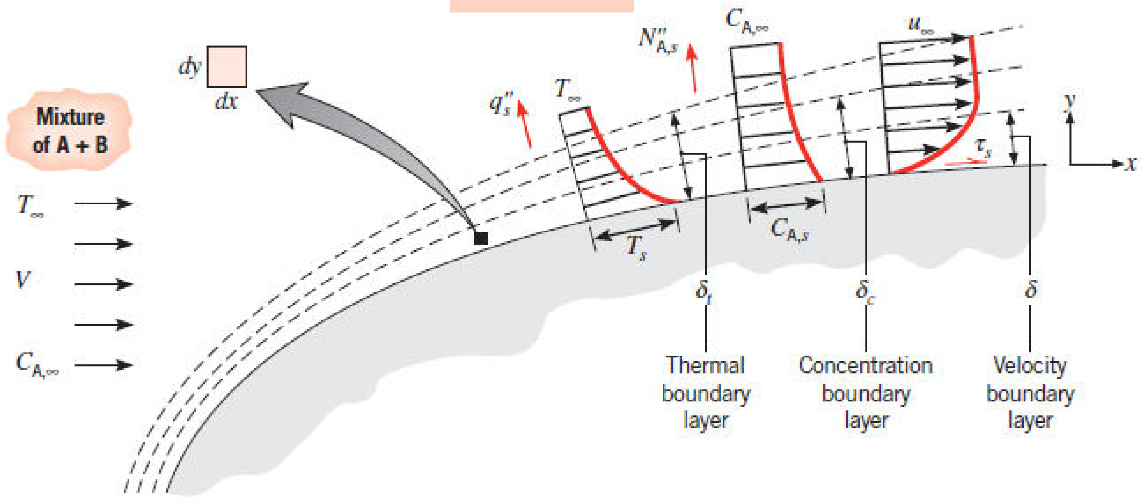
Assumptions involved in boundary layer equations

- *Body forces are negligible*, there is *no thermal energy generation in the fluid* and the *flow is non-reacting*

- Boundary layer thicknesses are typically very small relative to the size of the object upon which they form, and the *x*-direction velocity, temperature gradients normal to object surface are very small

$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$$

- By neglecting the *x*-direction terms, the net shear stress, conduction heat flux, in the *x*-direction are negligible. Since the boundary layers are so thin, the *x*-direction pressure gradient within a boundary layer can be approximated as the free stream pressure gradient $\frac{\partial p}{\partial x} \approx \frac{dp_\infty}{dx}$



Development of the velocity, thermal, and concentration boundary layers for an arbitrary surface

Assumptions involved in boundary layer equations

- The form of $p_\infty(x)$ depends on the surface geometry and may be obtained from a separate consideration of flow conditions in the free stream where shear stresses are negligible. Hence pressure gradient is treated as known quantity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

-- 2D continuity equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

-- 2D momentum equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

-- 2D energy equation

- Analytical solutions of boundary layers involve complicated mathematics and detailed solutions can be obtained by finite difference or finite element techniques

- Wide array of situations of engineering relevance involve turbulent convective heat transfer which is mathematically and physically more complex than laminar convection

Boundary layer similarity: The normalized boundary layer equations

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

- The above equations are characterized by advection terms left hand side and diffusion terms on right hand side

- The above situation is described by low speed forced convection flows which are found in many engineering applications

- Implications of similarity may be developed in a rational manner by first non dimesionalizing the governing equations

Boundary layer similarity parameters

- The boundary layer equations are *normalized* by first defining dimensionless independent variables of the forms:

$$x^* \equiv \frac{x}{L} \qquad \text{and} \qquad y^* \equiv \frac{y}{L}$$

where *L* is a *characteristic length* for the surface of interest

- Dependent dimensionless variables may be defined as:

$$u^* \equiv \frac{u}{V} \qquad \text{and} \qquad v^* \equiv \frac{v}{V} \qquad\qquad T^* \equiv \frac{T - T_s}{T_\infty - T_s}$$

where *V* is the velocity upstream of the surface

- Dimensionless form of conservation equations can be developed using dimensionless variables and as a consequence two important dimensionless similarity parameters evolve: Reynolds number and Prandtl number
- Similarity parameters allow to extrapolate results of a surface experiencing a set of convective conditions to geometrically similar surfaces subjected to entirely different conditions
- As long as similarity parameters and dimensionless boundary conditions are same for two sets of conditions, the solution of differential equations for velocity and temperature will be identical

The boundary layer equations and their *y*-direction boundary conditions in nondimensional form

Boundary Layer	Conservation Equation	Boundary Conditions		Similarity Parameter(s)
		Wall	Free Stream	
Velocity	$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{dp^*}{dx^*}+\frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}}$	$u^*(x^*,0)=0$	$u^*(x^*,\infty)=\frac{u_\infty(x^*)}{V}$	$Re_L=\frac{VL}{\nu}$
Thermal	$u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{1}{Re_L\,Pr}\frac{\partial^2 T^*}{\partial y^{*2}}$	$T^*(x^*,0)=0$	$T^*(x^*,\infty)=1$	$Re_L,\ Pr=\frac{\nu}{\alpha}$

- The momentum equation suggests that though conditions in velocity boundary layer depends upon fluid properties ρ, μ, velocity, V and length scale L, this dependence may be simplified by grouping variables in the form of Reynolds number

- The functional form of solution of the differential equation is given by:

$$u^*=f\left(x^*,y^*,Re_L,\frac{dp^*}{dx^*}\right)$$

- Since the pressure distribution *p**(*x**) depends on the surface geometry and may be obtained independently by considering flow conditions in the free stream, the appearance of *dp**/*dx** momentum equation represents the influence of geometry on the velocity distribution

Non dimensional correlation of momentum equation

The shear stress at the surface, $y^* = 0$, may be expressed as:

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \left(\frac{\mu V}{L}\right) \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0}$$

Based on the above relation, friction coefficient can be expressed as:

$$C_f = \frac{\tau_s}{\rho V^2/2} = \frac{2}{Re_L} \frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0}$$

Since we know that :

$$\frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0} = f\left(x^*, Re_L, \frac{dp^*}{dx^*}\right)$$

For a prescribed geometry:

$$C_f = \frac{2}{Re_L} f(x^*, Re_L)$$

Friction coefficient, a dimensionless parameter of considerable importance to the engineer, may be expressed exclusively in terms of a dimensionless space coordinate and the Reynolds number

Non dimensional correlations of energy equation

- Intuitively, we expect that convective heat transfer coefficient, *h* depends on the fluid properties (*k*, c_p, μ, and ρ), the fluid velocity *V*, the length scale *L*, and the surface geometry

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L\, Pr} \frac{\partial^2 T^*}{\partial y^{*2}}$$

- Based on the above equation, the expression for temperature can be :

$$T^* = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right)$$

- The dependence on *dp*/dx** originates from the influence of the geometry on the fluid motion (*u** and *v**), which, in turn, affects the thermal conditions

$$h = -\frac{k_f}{L}\frac{(T_\infty - T_s)}{(T_s - T_\infty)}\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = +\frac{k_f}{L}\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$Nu \equiv \frac{hL}{k_f} = +\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$Nu = f(x^*, Re_L, Pr)$$

Nusselt number to thermal boundary layer is what friction to velocity boundary layer. It can be computed for different fluids and different values of V and L

$$\overline{Nu} = \frac{\bar{h}L}{k_f} = f(Re_L, Pr)$$

Physical interpretation of dimensionless Reynolds number

- Dimensionless parameters have physical interpretations that relate to conditions in the flow, not only for boundary layers but also for other flow types, such as the internal flows

- *Reynolds number* Re_L, may be interpreted as the *ratio of inertia to viscous forces* in a region of characteristic dimension *L.* Inertia forces are associated with increase in momentum of a moving fluid

- Inertia forces (per unit mass) are of the form $u\, \partial u/\partial x$. *The order of magnitude approximation gives* $F_I \sim V^2/L$

- Similarly, the net shear force (per unit mass) is $\mu(\partial u/\partial y)$ and is approximated as $F_s \approx \mu V/L^2$

$$\frac{F_I}{F_s} \approx \frac{\rho V^2/L}{\mu V/L^2} = \frac{\rho V L}{\mu} = Re_L$$

- *Re_L determines the existence of laminar or turbulent flow. The magnitude Re_L influences the velocity boundary layer thickness δ . With increasing Re_L at a fixed location on a surface, viscous forces become less influential relative to inertia forces. Hence the effects of viscosity do not penetrate as far into the free stream, and the value of δ diminishes*

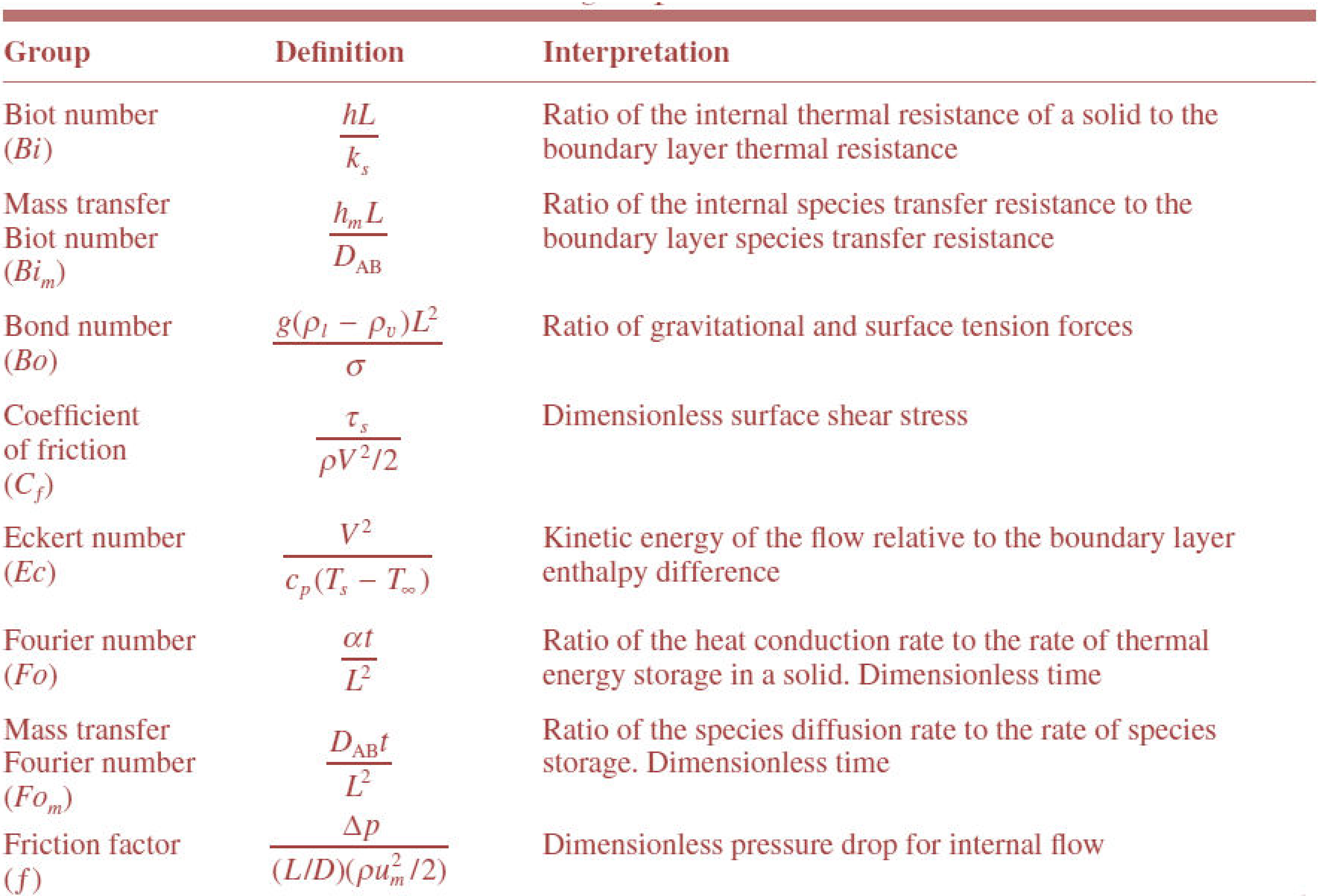
Physical interpretation of dimensionless Prandtl number

- The *Prandtl number, Pr* is defined as the ratio of the kinematic viscosity, also referred to as the momentum diffusivity, ν, to the thermal diffusivity α.
- Value of *Pr* strongly influences the relative growth of the velocity and thermal boundary layers
- For laminar boundary layers (in which transport by diffusion is *not* overshadowed by turbulent mixing):

$$\frac{\delta}{\delta_t} \approx Pr^n$$

- Prandtl number of gases is near unity, in which case $\delta_t \approx \delta$. For oils of highly viscous nature : $Pr >> 1$ and $\delta_t << \delta$

Selected dimensionless groups of heat and mass transfer -- Reference purpose



Group	Definition	Interpretation
Biot number (Bi)	$\frac{hL}{k_s}$	Ratio of the internal thermal resistance of a solid to the boundary layer thermal resistance
Mass transfer Biot number (Bi_m)	$\frac{h_m L}{D_{AB}}$	Ratio of the internal species transfer resistance to the boundary layer species transfer resistance
Bond number (Bo)	$\frac{g(\rho_l - \rho_v)L^2}{\sigma}$	Ratio of gravitational and surface tension forces
Coefficient of friction (C_f)	$\frac{\tau_s}{\rho V^2/2}$	Dimensionless surface shear stress
Eckert number (Ec)	$\frac{V^2}{c_p(T_s - T_\infty)}$	Kinetic energy of the flow relative to the boundary layer enthalpy difference
Fourier number (Fo)	$\frac{\alpha t}{L^2}$	Ratio of the heat conduction rate to the rate of thermal energy storage in a solid. Dimensionless time
Mass transfer Fourier number (Fo_m)	$\frac{D_{AB}t}{L^2}$	Ratio of the species diffusion rate to the rate of species storage. Dimensionless time
Friction factor (f)	$\frac{\Delta p}{(L/D)(\rho u_m^2/2)}$	Dimensionless pressure drop for internal flow

Contd----

Group	Definition	Interpretation
Grashof number (Gr_L)	$\frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$	Measure of the ratio of buoyancy forces to viscous forces
Colburn *j* factor (j_H)	$St\, Pr^{2/3}$	Dimensionless heat transfer coefficient
Colburn *j* factor (j_m)	$St_m\, Sc^{2/3}$	Dimensionless mass transfer coefficient
Jakob number (Ja)	$\frac{c_p(T_s - T_{\text{sat}})}{h_{fg}}$	Ratio of sensible to latent energy absorbed during liquid–vapor phase change
Lewis number (Le)	$\frac{\alpha}{D_{\text{AB}}}$	Ratio of the thermal and mass diffusivities
Mach number (Ma)	$\frac{V}{a}$	Ratio of velocity to speed of sound
Nusselt number (Nu_L)	$\frac{hL}{k_f}$	Ratio of convection to pure conduction heat transfer
Peclet number (Pe_L)	$\frac{VL}{\alpha} = Re_L\, Pr$	Ratio of advection to conduction heat transfer rates

Contd----

Group	Definition	Interpretation
Prandtl number (Pr)	$\frac{c_p\mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities
Reynolds number (Re_L)	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces
Schmidt number (Sc)	$\frac{\nu}{D_{\mathrm{AB}}}$	Ratio of the momentum and mass diffusivities
Sherwood number (Sh_L)	$\frac{h_m L}{D_{\mathrm{AB}}}$	Ratio of convection to pure diffusion mass transfer
Stanton number (St)	$\frac{h}{\rho V c_p} = \frac{Nu_L}{Re_L\ Pr}$	Modified Nusselt number
Mass transfer Stanton number (St_m)	$\frac{h_m}{V} = \frac{Sh_L}{Re_L\ Sc}$	Modified Sherwood number
Weber number (We)	$\frac{\rho V^2 L}{\sigma}$	Ratio of inertia to surface tension forces

Boundary layer analogies

C_f and Nu are of primary interest to convection heat transfer problems in this course

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L\, Pr}\frac{\partial^2 T^*}{\partial y^{*2}}$$

Fluid Flow

$$u^* = f\left(x^*, y^*, Re_L, \frac{dp^*}{dx^*}\right)$$

$$C_f = \frac{2}{Re_L}\frac{\partial u^*}{\partial y^*}\bigg|_{y^*=0}$$

$$C_f = \frac{2}{Re_L} f(x^*, Re_L)$$

Heat Transfer

$$T^* = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right)$$

$$Nu = \frac{hL}{k} = +\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0}$$

$$Nu = f(x^*, Re_L, Pr)$$

$$\overline{Nu} = f(Re_L, Pr)$$

- *If two or more processes are governed by dimensionless equations of the same form, the processes are said to be analogous. The above equations for a particular geometry are interchangeable*

HEAT TRANSFER - 80329

MODULE III
FORCED CONVECTION

External flows – Solutions of convection equations for Flat plates,

Part 5

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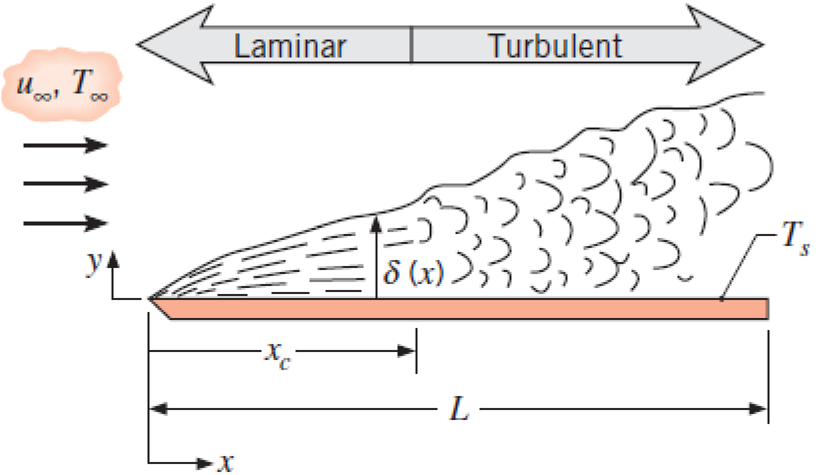
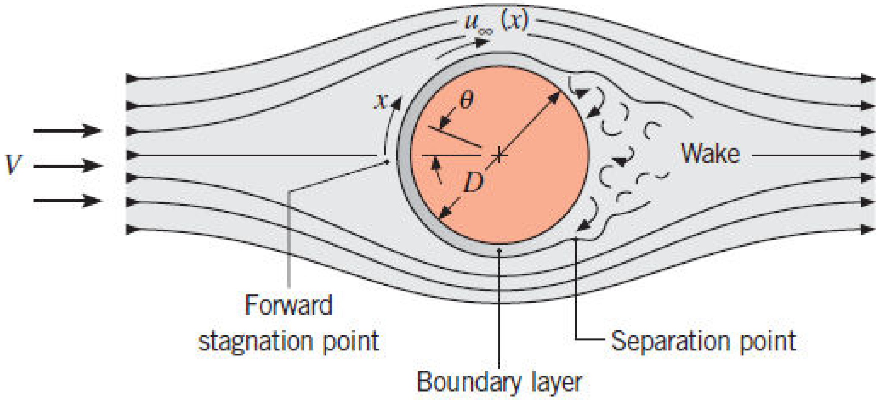
Associate Professor

Assumptions involved in deriving numerical relations for external flow

- In this module we focus on the problem of computing heat transfer rates to or from a surface in *external flow.* In such a flow boundary layers develop freely, without constraints imposed by adjacent surfaces
- Examples of external flow include fluid motion over a flat plate (inclined or parallel to the free stream velocity) and flow over curved surfaces such as a sphere, cylinder, airfoil, or turbine blade
- The derivations here are confined to problems of low-speed, forced convection with no phase change occurring within the fluid
- In forced convection, the relative motion between the fluid and the surface is maintained by external means, such as a fan or a pump, and not by buoyancy forces due to temperature gradients in the fluid





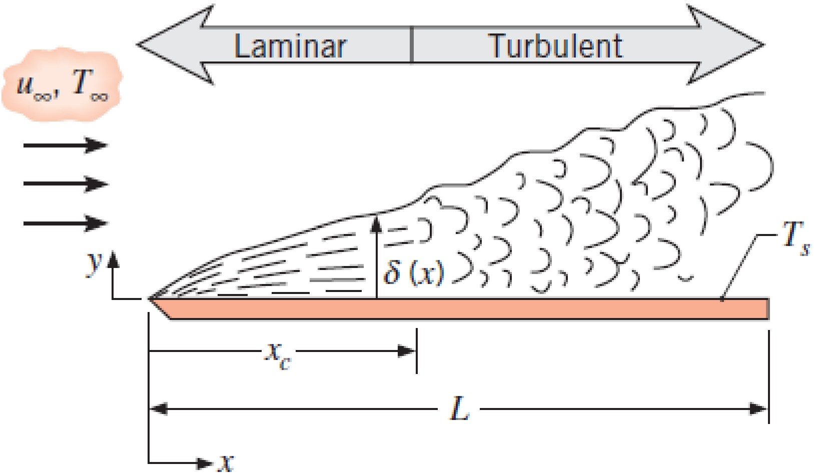
Assumptions involved in deriving numerical relations for external flow

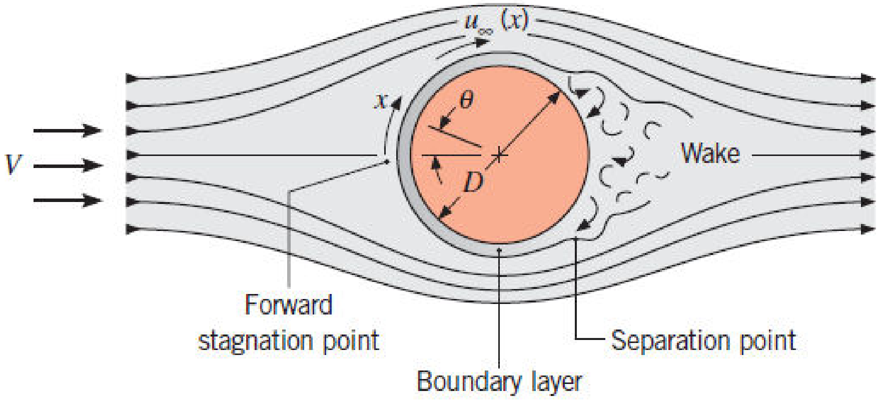
- The primary objective is to determine convection coefficients for different flow geometries. In particular, we wish to obtain specific forms of the functions that represent these coefficients

$$\boxed{Nu_x = f(x^*, Re_x, Pr)}$$

$$\boxed{\overline{Nu}_x = f(Re_x, Pr)}$$

- The subscript *x* emphasizes conditions at a particular location on the surface. The overbar indicates an average from *x** = 0, where the boundary layer begins to develop, to the location of interest
- The theoretical approach involves solving the boundary layer equations for a particular geometry. For example, obtaining the temperature profile T* from such a solution





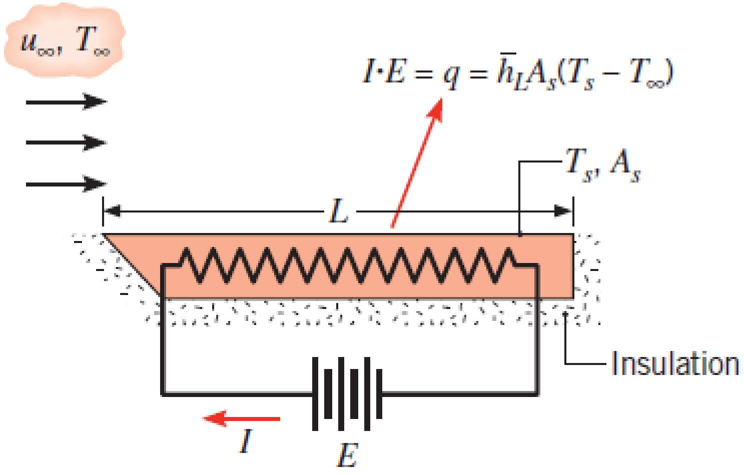
Experimental method to determine Nusselt number

- If a prescribed geometry, such as the flat plate in parallel flow, is heated electrically to maintain $T_s > T_\infty$, convection heat transfer occurs from the surface to the fluid

- It would be a simple matter to measure T_s and T_∞, as well as the electrical power, $E \cdot I$, which is equal to the total heat transfer rate q. The convection coefficient $\bar{h}_L$, which is an average associated with the entire plate, could then be computed from Newton's law of cooling

- From knowledge of the characteristic length L and the fluid properties, the Nusselt, Reynolds, and Prandtl numbers could be computed from their definitions

- *The foregoing procedure could be repeated for a variety of test conditions. We could vary the velocity u_∞ and the plate length L, as well as the nature of the fluid, using, for example, air, water, and engine oil, which have substantially different Prandtl numbers*

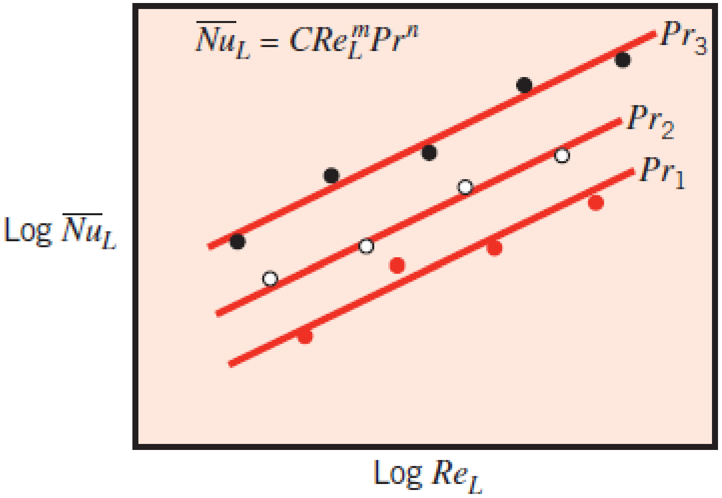


Experimental method to determine Nusselt number

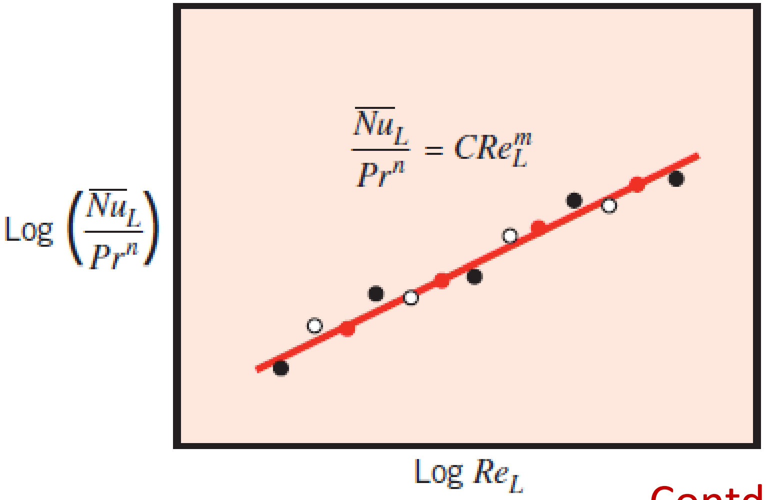
- By experimenting with many different values of the Nusselt numbers corresponding to a wide range of Reynolds and Prandtl numbers, the results could be plotted on a log–log scale, as shown in Fig. a
- The results associated with a given fluid, and hence a fixed Prandtl number, fall close to a straight line, indicating a power law dependence of the Nusselt number on the Reynolds number. Considering all the fluids, the data may then be represented by an algebraic expression of the form:

$$\overline{Nu}_L = C\, Re_L^m\, Pr^n$$

- Since the values of *C*, *m*, and *n* are often independent of the nature of the fluid, the family of straight lines corresponding to different Prandtl numbers can be collapsed to a single line by plotting the results in terms of the ratio, $\tilde{N}_L / Pr^n$, as shown in Fig. b



(a)



(b)

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Experimental method to determine Nusselt number

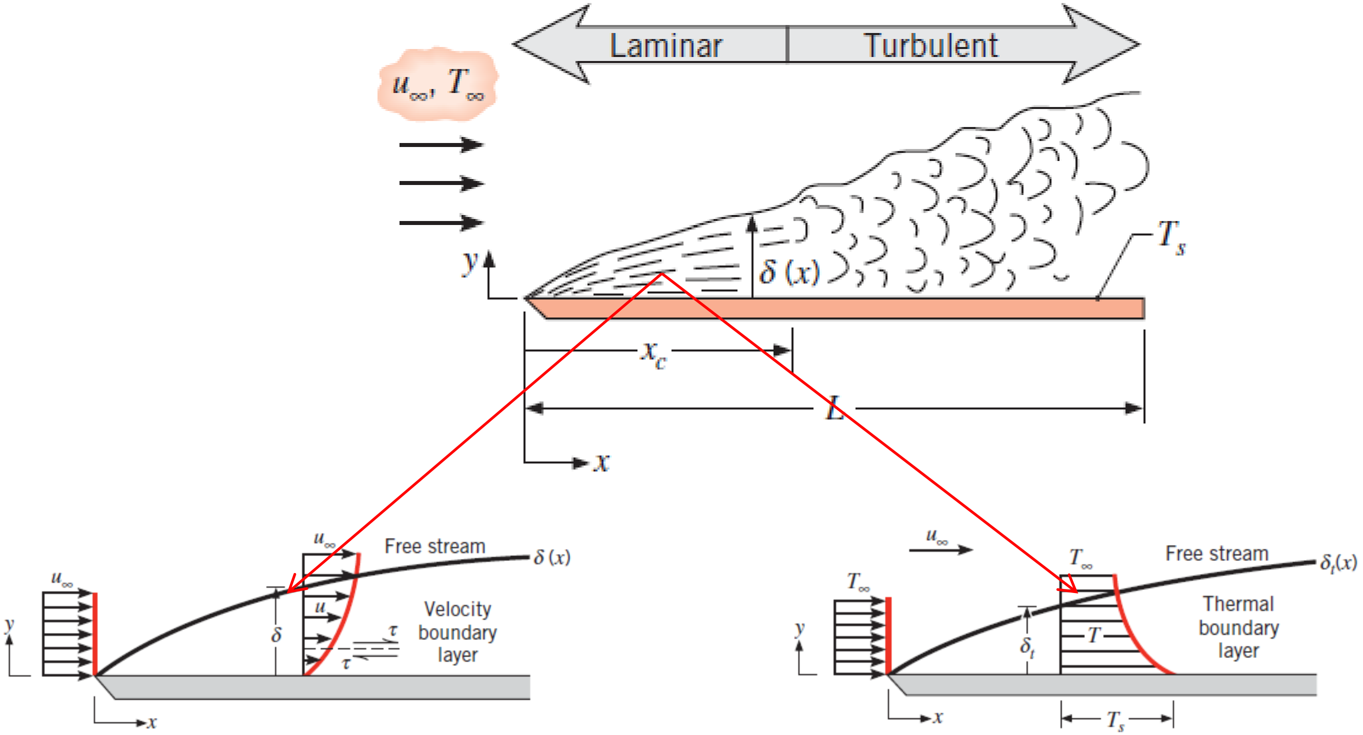
$$\overline{Nu}_L = C\, Re_L^m\, Pr^n$$

- The above numerical expression for average Nusselt number will be invoked for many special cases, and it is important to note that the assumption of *constant fluid properties* is often implicit in the results
- As the fluid properties vary with temperature across the boundary layer and this variation can certainly influence the heat transfer rate . Hence such variation is catered for in the above numerical expression in two ways as below:
- In first, average Nusselt number is used with all properties evaluated at a mean boundary layer temperature T_f, termed the *film temperature*

$$T_f \equiv \frac{T_s + T_\infty}{2}$$

- The alternate method is to evaluate all properties at T_∞ and to multiply the right-hand side of the above Nusselt number Equation by an additional parameter to account for the property variations.
- The parameter is commonly of the form $(Pr_\infty/Pr_s)^r$ or $(\mu_\infty/\mu_s)^r$, where the subscripts ∞ and *s* designate evaluation of the properties at the free stream and surface temperatures, respectively

Derivation for a laminar flow over isothermal flat plate: Methodology



- The first step is analytical determination of the velocity and temperature distributions in the laminar boundary layers
- From the knowledge of these distributions, expressions for local, average friction coefficients and Nusselt numbers can be determined

Contd----

Derivation for a laminar flow over isothermal flat plate : A similarity solution

- The major convection parameters may be obtained by solving the appropriate form of the boundary layer equations

- *Assuming steady, incompressible, laminar flow with constant fluid properties, negligible viscous dissipation and recognizing that dp/dx = 0, the boundary layer equations can be formulated as below:*

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

- Solution of these equations is simplified by the fact that for constant properties, conditions in the velocity (hydrodynamic) boundary layer are independent of temperature and species concentration

- Once the hydrodynamic problem is solved momentum and energy equations can be solved easily

Hydrodynamic solution to the flat plate

- *Hydrodynamic solution for the flat plate problem is obtained by the method of Blasius, a German Engineer who invented this solution. The first step in this method is to define a stream function Ψ (x, y) such that:*

$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x}$$

- Based on the stream function assumption, continuity equation , is automatically satisfied and no longer needed.

- Next new dependent and independent variables, f and η, respectively, are then defined as follows:

$$f(\eta) \equiv \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}}$$

$$\eta \equiv y \sqrt{u_\infty / \nu x}$$

- *This introduction of new independent variable,η called similarity variable allows transformation of two partial differential equations (momentum and energy equations) into a single ordinary differential equation*

- **Blasius reasoned that the nondimensional velocity profile *u/u_∞* should remain unchanged when plotted against the nondimensional distance *y/δ*, where δ is the thickness of the local velocity boundary layer at a given *x***

Contd----

Hydrodynamic solution to the flat plate

- The Blasius solution is termed a *similarity solution,* and η is a *similarity variable.* This terminology is used because, despite growth of the boundary layer with distance *x* from the leading edge, the velocity profile u/u_∞ remains *geometrically similar*

$$\frac{u}{u_\infty} = \phi\left(\frac{y}{\delta}\right) \Rightarrow \frac{u}{u_\infty} = \phi(\eta)$$

- *Since it has been proved by Stokes earlier that boundary layer thickness δ is proportional to* $\sqrt{\frac{\nu x}{u_\infty}}$ it leads to expression for similarity variable η

$$f(\eta) \equiv \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}}$$

$$\eta \equiv y\sqrt{u_\infty / \nu x}$$

- As per definition of stream function the velocity components u and v are:

$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \sqrt{\frac{\nu x}{u_\infty}}\frac{df}{d\eta}\sqrt{\frac{u_\infty}{\nu x}} = u_\infty \frac{df}{d\eta} \qquad v = -\frac{\partial \psi}{\partial x} = -\left(u_\infty \sqrt{\frac{\nu x}{u_\infty}}\frac{\partial f}{\partial x} + \frac{u_\infty}{2}\sqrt{\frac{\nu}{u_\infty x}}f\right)$$

Contd----

Hydrodynamic solution to the flat plate

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \sqrt{\frac{\nu x}{u_\infty}}\frac{df}{d\eta}\sqrt{\frac{u_\infty}{\nu x}} = u_\infty \frac{df}{d\eta}$$

$$v = -\frac{\partial \psi}{\partial x} = -\left(u_\infty \sqrt{\frac{\nu x}{u_\infty}}\frac{\partial f}{\partial x} + \frac{u_\infty}{2}\sqrt{\frac{\nu}{u_\infty x}} f \right)$$

Velocity component *v* can be simplified as:

$$v = \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left(\eta \frac{df}{d\eta} - f \right)$$

By differentiating the velocity components, it may also be shown that:

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}\eta \frac{d^2 f}{d\eta^2}$$

$$\frac{\partial u}{\partial y} = u_\infty \sqrt{\frac{u_\infty}{\nu x}}\frac{d^2 f}{d\eta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_\infty^2}{\nu x}\frac{d^3 f}{d\eta^3}$$

Substituting the above terms in momentum equation :

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

Contd----

Hydrodynamic solution to the flat plate

Hence the modified momentum equation can be written as

$$2\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} = 0$$

Hence the hydrodynamic boundary layer problem is reduced to one of solving a nonlinear, third-order ordinary differential equation. The appropriate boundary conditions are:

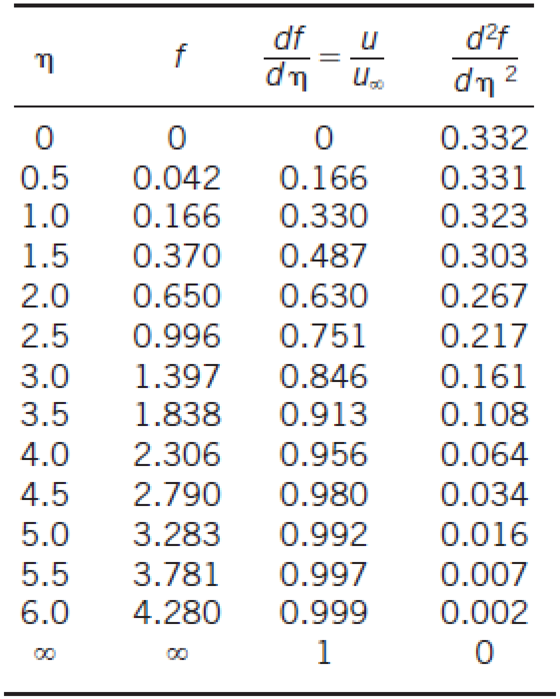
$$u(x,0) = v(x,0) = 0 \quad \text{and} \quad u(x,\infty) = u_\infty$$

In term of similarity variables the boundary conditions are:

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = f(0) = 0 \quad \text{and} \quad \left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 1$$

The solution to the differential equation, subject to the conditions above, may be obtained by a series expansion or by numerical integration

$$\boxed{\delta = \frac{5.0}{\sqrt{u_\infty/\nu x}} = \frac{5.0x}{\sqrt{Re_x}}}$$



η	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.5	0.042	0.166	0.331
1.0	0.166	0.330	0.323
1.5	0.370	0.487	0.303
2.0	0.650	0.630	0.267
2.5	0.996	0.751	0.217
3.0	1.397	0.846	0.161
3.5	1.838	0.913	0.108
4.0	2.306	0.956	0.064
4.5	2.790	0.980	0.034
5.0	3.283	0.992	0.016
5.5	3.781	0.997	0.007
6.0	4.280	0.999	0.002
∞	∞	1	0

Flat plate laminar boundary layer functions

Contd----

Hydrodynamic solution to the flat plate

$$\boxed{\delta = \frac{5.0}{\sqrt{u_\infty / \nu x}} = \frac{5.0x}{\sqrt{\mathrm{Re}_x}}}$$

- *The boundary layer thickness increases with increasing kinematic viscosity ν and with increasing distance from the leading edge but decreases with increasing free stream velocity . Hence a large free stream velocity will suppress the boundary layer and causes it to be thinner*

- The shear stress on the wall can be determined from its definition and the *∂u/ ∂y* relation:

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu u_\infty \sqrt{\frac{u_\infty}{\nu x}} \frac{d^2 f}{d\eta^2}\bigg|_{\eta=0}$$

Substituting the value of the second derivative of *f* at η = 0 from the table:

$$\boxed{\tau_w = 0.332 u_\infty \sqrt{\frac{\rho \mu u_\infty}{x}} = \frac{0.332 \rho u_\infty^2}{\sqrt{\mathrm{Re}_x}}}$$

Then the local skin friction coefficient becomes:

$$\boxed{C_{f,x} = \frac{\tau_w}{\rho \mathcal{V}^2/2} = \frac{\tau_w}{\rho u_\infty^2/2} = 0.664\ \mathrm{Re}_x^{-1/2}}$$

η	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.5	0.042	0.166	0.331
1.0	0.166	0.330	0.323
1.5	0.370	0.487	0.303
2.0	0.650	0.630	0.267
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5.5	3.781	0.997	0.007
6.0	4.280	0.999	0.002
∞	∞	1	0

Flat plate laminar boundary layer functions

Solution to the thermal energy equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

- From knowledge of conditions in the velocity boundary layer, the energy equation can now be solved
- By introducing a dimensionless temperature, T* and assuming a similarity solution of the form: T* = T (η)

$$T^* \equiv [(T - T_s)/(T_\infty - T_s)]$$

- The energy equation which originally is a partial differential equation reduces to:

$$\frac{d^2T^*}{d\eta^2} + \frac{Pr}{2} f \frac{dT^*}{d\eta} = 0$$

- Note the dependence of the thermal solution on hydrodynamic conditions through appearance of the variable *f* in the above equation. The appropriate boundary conditions are:

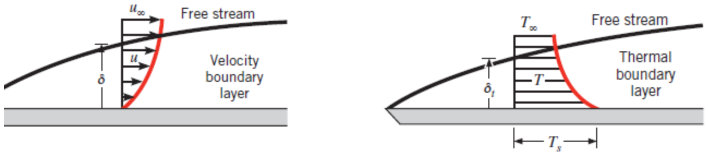
$$T^*(0) = 0 \quad \text{and} \quad T^*(\infty) = 1$$

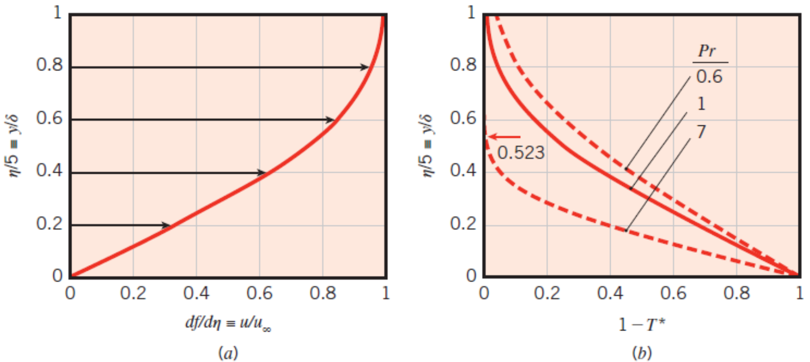
Solution to the thermal energy equation

$$\frac{d^2T^*}{d\eta^2} + \frac{Pr}{2} f \frac{dT^*}{d\eta} = 0 \qquad T^*(0) = 0 \quad \text{and} \quad T^*(\infty) = 1$$

- Energy equation may be solved by numerical integration for different values of the Prandtl number; representative temperature distributions are shown below:
- The temperature distribution is identical to the velocity distribution for *Pr* = 1. Thermal effects penetrate farther into the velocity boundary layer with decreasing Prandtl number and transcend the velocity boundary layer for *Pr* < 1
- A practical consequence of this solution is that, for *Pr* ≥ 0.6, results for the surface temperature gradient *dT*/d*η @ η =0 may be correlated by the following relation

$$\left.\frac{dT^*}{d\eta}\right|_{\eta=0} = 0.332\,Pr^{1/3}$$





(*a*) (*b*)

Similarity solution for laminar flow over an isothermal plate. (*a*) The *x*- component of the velocity. (*b*) Temperature distributions for *Pr* = 0.6, 1, and 7

Solution to the thermal energy equation

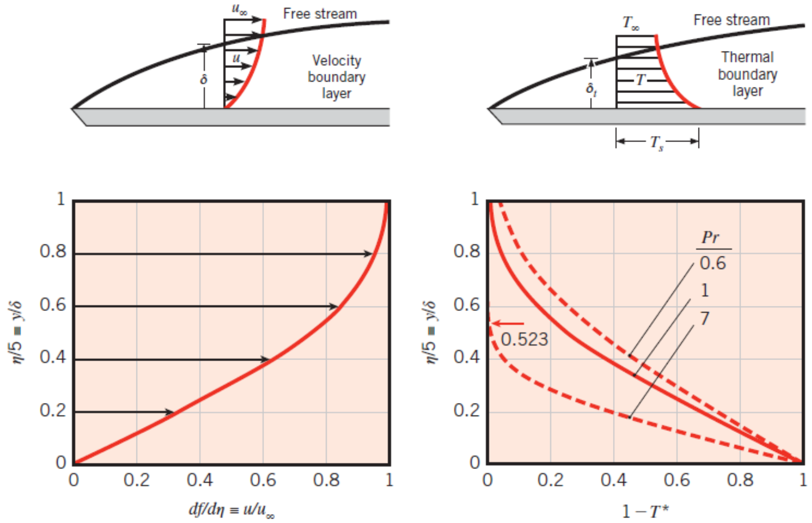
$$h_x = \frac{q_s''}{T_s - T_\infty} = -\frac{T_\infty - T_s}{T_s - T_\infty} k \frac{\partial T^*}{\partial y}\bigg|_{y=0}$$

$$h_x = k\left(\frac{u_\infty}{\nu x}\right)^{1/2} \frac{dT^*}{d\eta}\bigg|_{\eta=0}$$

$$Nu_x \equiv \frac{h_x x}{k} = 0.332\, Re_x^{1/2} Pr^{1/3} \qquad Pr \gtrsim 0.6$$

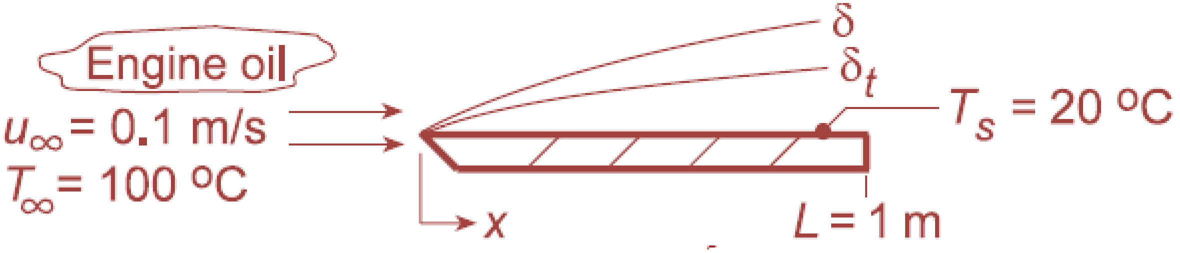
The ratio of the velocity to thermal boundary layer thickness is

$$\boxed{\frac{\delta}{\delta_t} \approx Pr^{1/3}}$$



(a) (b)

Problem: Engine oil at 100°C and a velocity of 0.1 m/s flows over both surfaces of a 1m-long flat plate maintained at 20°C. Determine a)The velocity and thermal boundary layer thicknesses at the trailing edge b) The local heat flux and surface shear stress at the trailing edge



PROPERTIES: *Table A.5*, Engine Oil ($T_f = 333$ K): $\rho = 864$ kg/m^3, $\nu = 86.1 \times 10^{-6}$ m^2/s, k = 0.140 W/m·K, Pr = 1081.

Solution: : Calculate the Reynolds number to determine nature of the flow

$$\mathrm{Re}_L = \frac{u_\infty L}{\nu} = \frac{0.1\,\mathrm{m/s} \times 1\,\mathrm{m}}{86.1 \times 10^{-6}\,\mathrm{m^2/s}} = 1161$$

At x = L the flow is laminar, hence boundary layer thickness can be calculated as:

$$\delta = 5L\,\mathrm{Re}_L^{-1/2} = 5(1\,\mathrm{m})(1161)^{-1/2} = 0.147\,\mathrm{m}$$

$$\delta_t = \delta\,\mathrm{Pr}^{-1/3} = 0.147\,\mathrm{m}(1081)^{-1/3} = 0.0143\,\mathrm{m}$$

The velocity and thermal boundary layers are given by expressions:

$$\delta = 5L\,\mathrm{Re}_L^{-1/2} = 5(1\,\mathrm{m})(1161)^{-1/2} = 0.147\,\mathrm{m}$$

$$\delta_t = \delta\,\mathrm{Pr}^{-1/3} = 0.147\,\mathrm{m}(1081)^{-1/3} = 0.0143\,\mathrm{m}$$

The local and convective heat flux are given by:

$$h_L = \frac{k}{L} 0.332\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3} = \frac{0.140\,\mathrm{W/m\cdot K}}{1\,\mathrm{m}} 0.332(1161)^{1/2}(1081)^{1/3} = 16.25\,\mathrm{W/m^2\cdot K}$$

$$q''_x = h_L\left(T_s - T_\infty\right) = 16.25\,\mathrm{W/m^2\cdot K}(20-100)^\circ\,\mathrm{C} = -1300\,\mathrm{W/m^2}$$

Contd----

Also, the local shear stress is:

$$\tau_{s,L} = \frac{\rho u_\infty^2}{2} 0.664\, Re_L^{-1/2} = \frac{864\,kg/m^3}{2}(0.1\,m/s)^2\, 0.664(1161)^{-1/2}$$

$$\tau_{s,L} = 0.0842\,kg/m \cdot s^2 = 0.0842\,N/m^2$$

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MODULE III
FORCED CONVECTION

External flows – Convective heat transfer correlations for Cylinders and Spheres

Part 6

Dr. Satish Idury
Associate Professor

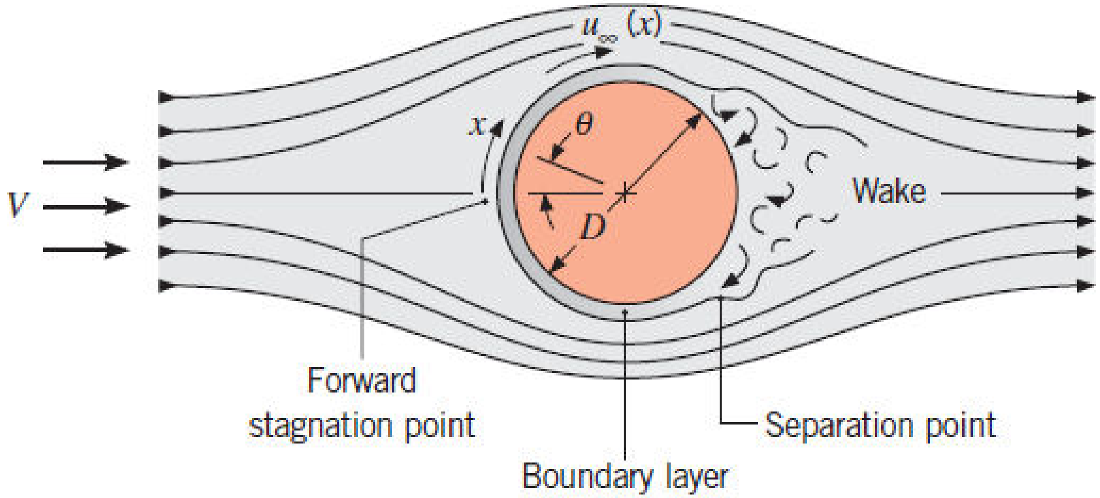
The cylinder in a cross flow

- As shown in Fig. the free stream fluid is brought to rest at the *forward stagnation point,* with an accompanying rise in pressure

- From this forward stagnation point, the pressure decreases with increasing *x*, the streamline coordinate, and the boundary layer develops under the influence of a *favorable pressure gradient* ($dp/dx < 0$)

- The pressure will eventually reach a minimum, and toward the rear of the cylinder further boundary layer development occurs in the presence of an *adverse pressure gradient* ($dp/dx > 0$)

- Unlike conditions for the flat plate in parallel flow, upstream velocity, V differ, with u_∞ now depending on the distance *x* from the stagnation point

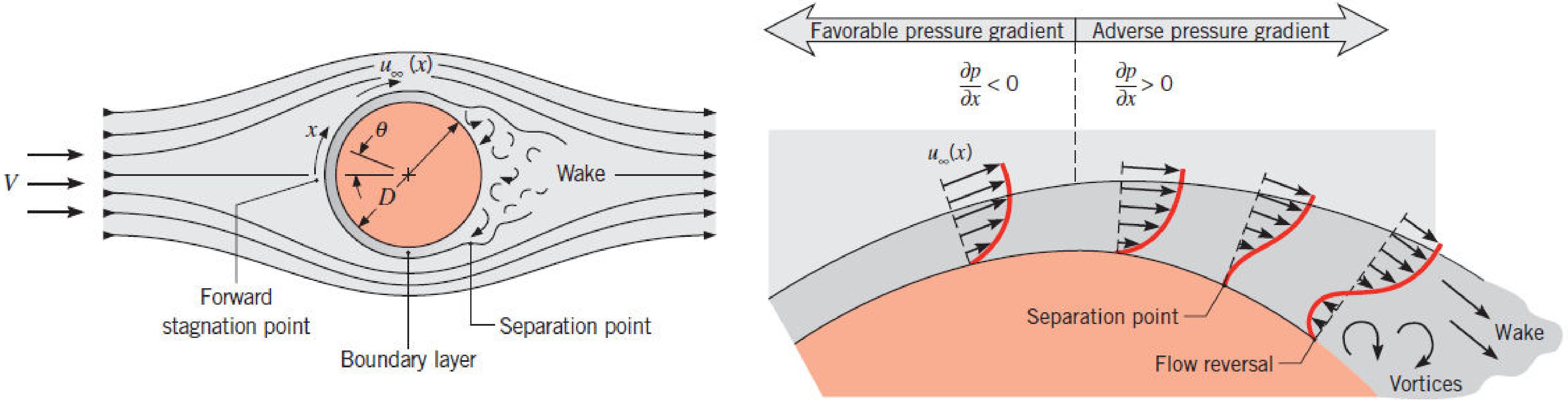


The cylinder in a cross flow

- From Euler's equation for an Inviscid flow, $u_\infty(x)$ must exhibit behavior opposite to that of $p(x)$. That is, from $u_\infty = 0$ at the stagnation point, the fluid accelerates because of the favorable pressure gradient ($du_\infty/dx > 0$ when $dp/dx < 0$), reaches a maximum velocity when $dp/dx = 0$, and decelerates because of the adverse pressure gradient ($du_\infty/dx < 0$) when $dp/dx > 0$)

- As the fluid decelerates, the velocity gradient at the surface, $\partial u/\partial y$ *at* $y=0$, eventually becomes zero. At this location, termed the *separation point,* fluid near the surface lacks sufficient momentum to overcome the pressure gradient, and continued downstream movement is impossible

- Since the oncoming fluid also precludes flow back upstream, *boundary layer separation* must occur. This is a condition for which the boundary layer detaches from the surface, and a *wake* is formed in the downstream region. Flow in this region is characterized by vortex formation and is highly irregular



Boundary layer formation and separation on a circular cylinder in cross flow

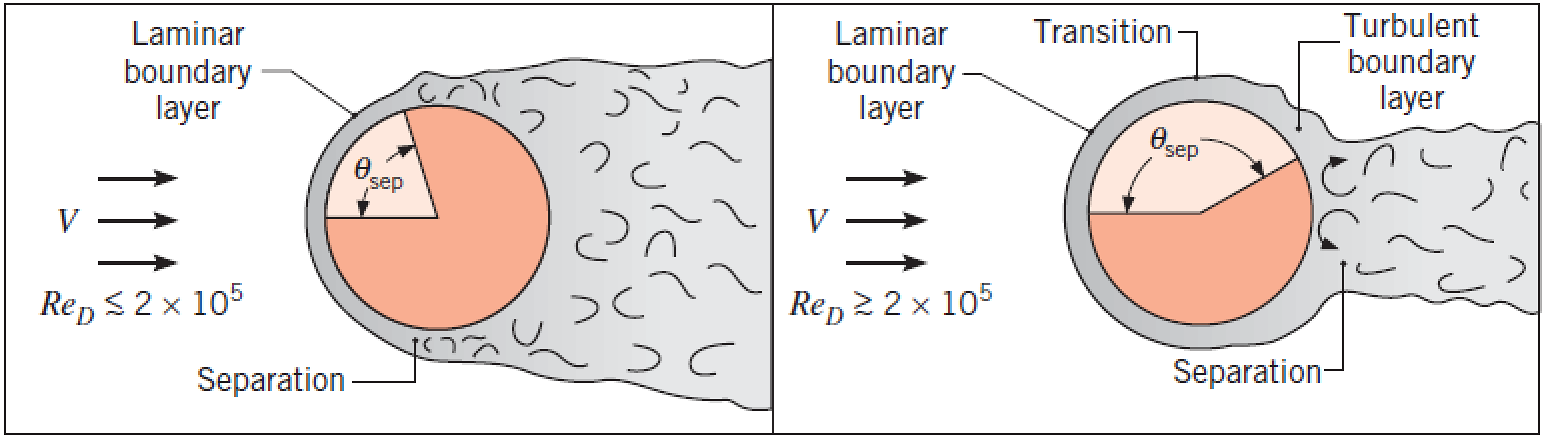
The cylinder in a cross flow

- The occurrence of *boundary layer transition,* which depends on the Reynolds number, strongly influences the position of the separation point. For the circular cylinder the characteristic length is the diameter, and the Reynolds number is defined as:

$$Re_D \equiv \frac{\rho V D}{\mu} = \frac{V D}{\nu}$$

- Since the momentum of fluid in a turbulent boundary layer is larger than in the laminar boundary layer, it is reasonable to expect transition to delay the occurrence of separation

- If $Re_D \leq 2 \times 10^5$, the boundary layer is laminar, and separation occurs at θ ≈ 80°). However, if $Re_D \geq 2 \times 10^5$, boundary layer transition occurs, and separation is delayed to θ ≈ 140°



The effect of turbulence on separation

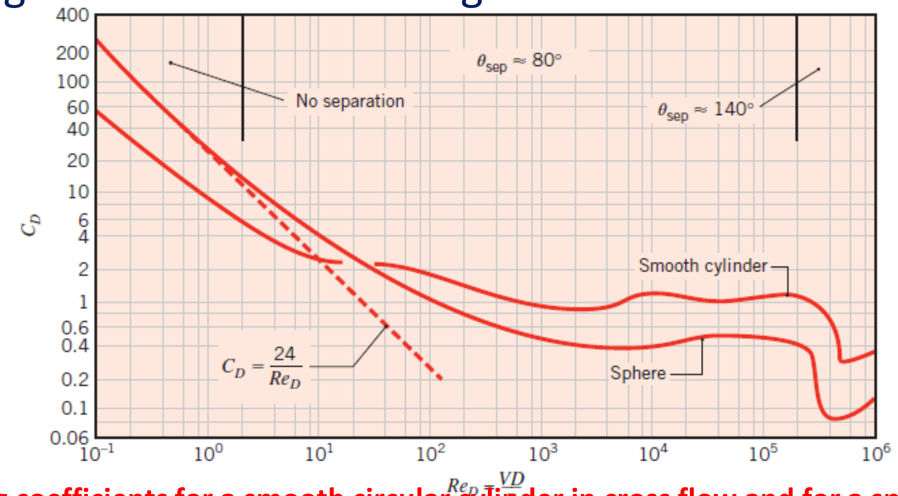
The cylinder in a cross flow

- Boundary layer transition processes strongly influence the drag force, F_D, acting on the cylinder. This force has two components, one of which is due to the boundary layer surface shear stress (*friction drag*). The other component is due to a pressure differential in the flow direction resulting from formation of the wake (*form, or pressure, drag*). A dimensionless *drag coefficient* C_D may be defined as

$$C_D \equiv \frac{F_D}{A_f(\rho V^2/2)}$$

A_f is the cylinder frontal area (the area projected perpendicular to the free stream velocity)

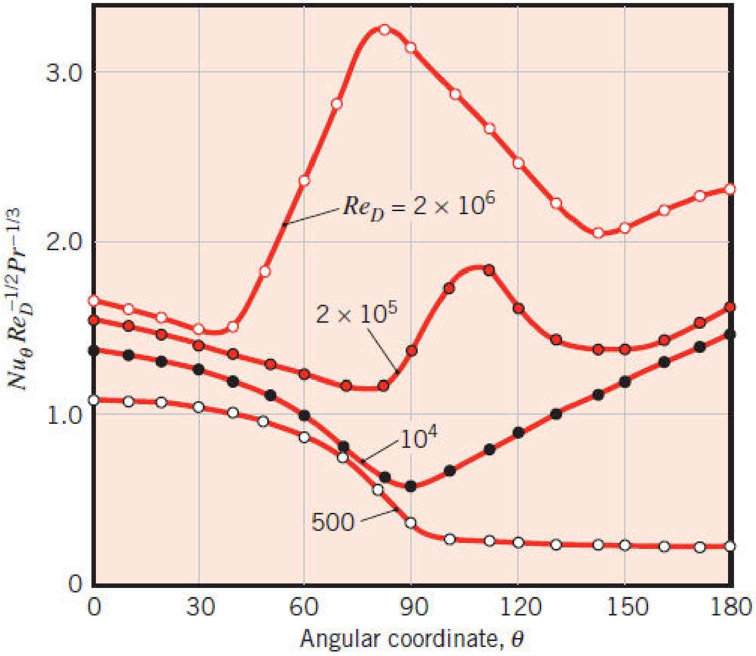
- The large reduction in C_D that occurs for $Re_D \geq 2 \times 10^5$ is due to boundary layer transition, which delays separation, thereby reducing the extent of the wake region and the magnitude of the form drag



Drag coefficients for a smooth circular cylinder in cross flow and for a sphere. Boundary layer separation angles are for a cylinder

The cylinder in a cross flow – Convection heat transfer correlation

- Experimental results for the variation of the local Nusselt number with θ are shown in Fig. for the cylinder in a cross flow of air . Nusselt number strongly varies with boundary layer development on the surface
- For $Re_D \leq 10^5$,starting at the stagnation point, Nu_θ decreases with increasing θ as a result of laminar boundary layer development. However, a minimum is reached at θ ≈ 80°, where separation occurs and Nu_θ increases with θ due to mixing associated with vortex formation in the wake
- For $Re_D \geq 10^5$, variation of Nusselt number is characterized by two minima. The increase in Nu_θ with increasing ReD is due to corresponding reduction in boundary layer thickness



Local Nusselt number for airflow normal to circular cylinder

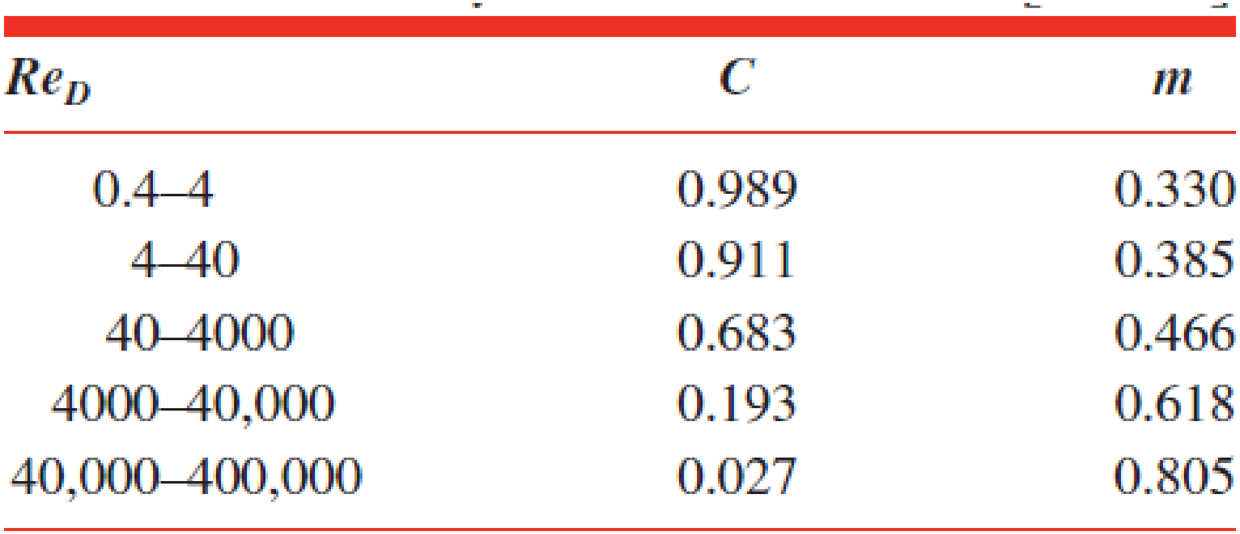
The cylinder in a cross flow – Correlation relations

At the forward stagnation point for Pr ≥ 0.6, boundary layer analysis yields an expression:

$$Nu_D(\theta = 0) = 1.15\, Re_D^{1/2}\, Pr^{1/3}$$

An empirical correlation due to Hilpert that has been modified to account for fluids of various Prandtl numbers for Pr ≥0.7 : The above equation is applicable for cylinders as well as non-circular cross sections. In this equation all the properties are evaluated at film temperature

$$\overline{Nu}_D \equiv \frac{\bar{h}D}{k} = C\, Re_D^m\, Pr^{1/3}$$



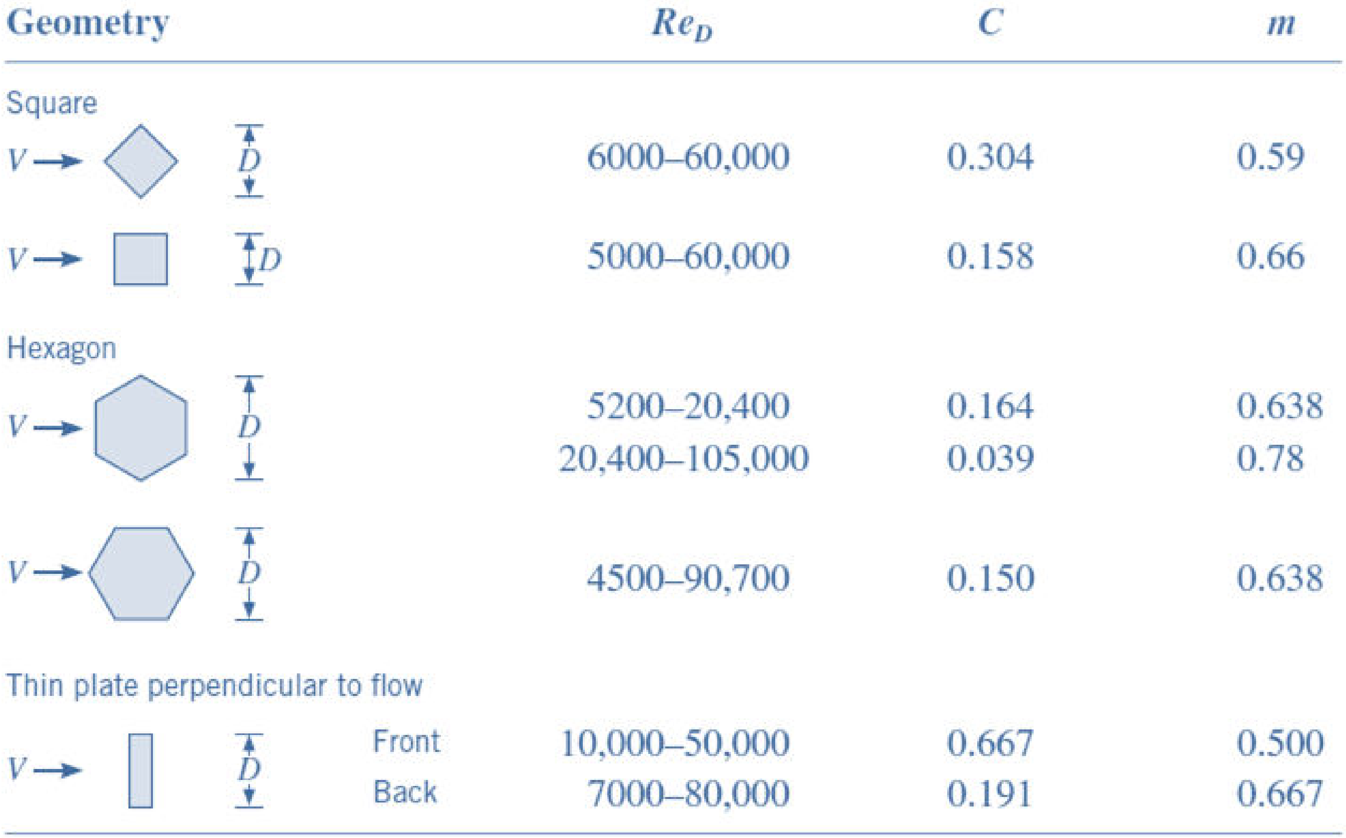
Re_D	C	m
0.4–4	0.989	0.330
4–40	0.911	0.385
40–4000	0.683	0.466
4000–40,000	0.193	0.618
40,000–400,000	0.027	0.805

The cylinder in a cross flow – Correlation relations

Other correlations have been suggested for the circular cylinder in cross flow where all properties are evaluated at T_∞ except Pr_s which is evaluated at $T_{s.}$. If $Pr \leq 10$, $n = 0.37$, If $Pr \geq 10$, $n = 0.36$

$$\overline{Nu}_D = C\,Re_D^m\,Pr^n\left(\frac{Pr}{Pr_s}\right)^{1/4}$$

$$\left[\begin{array}{l} 0.7 \lesssim Pr \lesssim 500 \\ 1 \lesssim Re_D \lesssim 10^6 \end{array}\right]$$



Geometry		Re_D	C	m
Square				
$V \rightarrow$ ◇ D		6000–60,000	0.304	0.59
$V \rightarrow$ □ D		5000–60,000	0.158	0.66
Hexagon				
$V \rightarrow$ ⬢ D		5200–20,400	0.164	0.638
		20,400–105,000	0.039	0.78
$V \rightarrow$ ⬣ D		4500–90,700	0.150	0.638
Thin plate perpendicular to flow				
$V \rightarrow$ ▯ D	Front	10,000–50,000	0.667	0.500
	Back	7000–80,000	0.191	0.667

The cylinder in a cross flow – Correlation relations

Churchill and Bernstein have proposed a single comprehensive equation that covers the entire range of Re_D *for which data are available, as well as a wide range of Pr. The equation is recommended for all* Re_D*,* $Pr \geq 0.2$ *and has the form:*

$$\overline{Nu}_D = 0.3 + \frac{0.62\, Re_D^{1/2}\, Pr^{1/3}}{\left[1 + (0.4/Pr)^{2/3}\right]^{1/4}} \left[1 + \left(\frac{Re_D}{282{,}000}\right)^{5/8}\right]^{4/5}$$

Each correlation is reasonable over a certain range of conditions, but for most engineering calculations one should not expect accuracy to much better than 20%.

Sphere – Correlation relations

Boundary layer effects associated with flow over a sphere are much like those for the circular cylinder, with transition and separation playing prominent roles. All the properties except μ_s are evaluated at T_∞

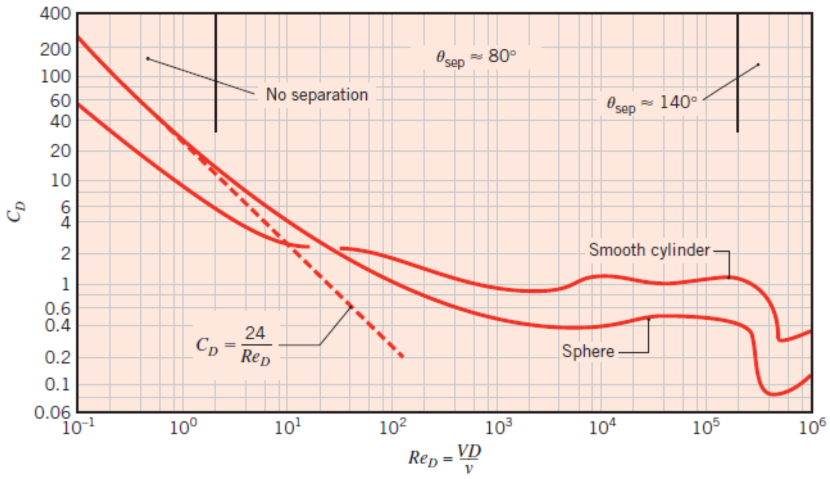
$$\overline{Nu}_D = 2 + (0.4\,Re_D^{1/2} + 0.06\,Re_D^{2/3})Pr^{0.4}\left(\frac{\mu}{\mu_s}\right)^{1/4}$$

$$\begin{bmatrix} 0.71 \lesssim Pr \lesssim 380 \\ 3.5 \lesssim Re_D \lesssim 7.6\times 10^4 \\ 1.0 \lesssim (\mu/\mu_s) \lesssim 3.2 \end{bmatrix}$$

In the limit of very small Reynolds numbers (creeping flow), the coefficient is inversely proportional to the Reynolds number and the specific relation is termed Stokes law

A special case of convection heat and mass transfer from spheres relates to transport from freely falling liquid drops is given by relation:

$$\overline{Nu}_D = 2 + 0.6\,Re_D^{1/2}\,Pr^{1/3}$$



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MODULE III
FORCED CONVECTION

MALLA REDDY ENGINEERING COLLEGE

MREC

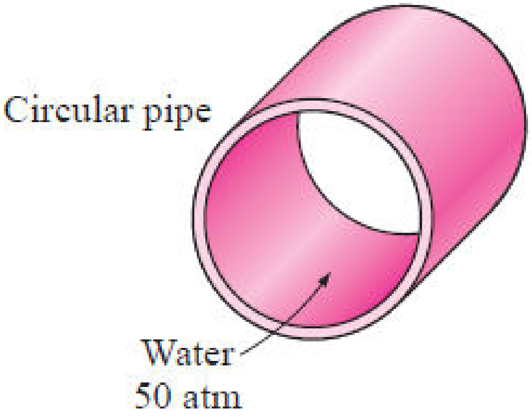
Internal flows – Concepts of hydrodynamic and thermal entry lengths, Flow through pipes and ducts and their empirical relations

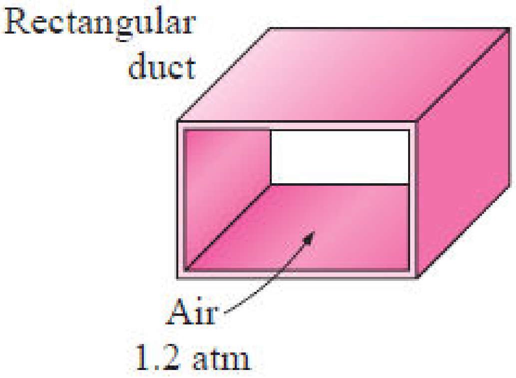
Part 7

Dr. Satish Idury
Associate Professor

Introduction- Internal flow

- Most fluids, especially liquids, are transported in circular pipes. This is because pipes with a circular cross section can withstand large pressure differences between the inside and the outside without undergoing any distortion
- Noncircular pipes are usually used in applications such as the heating and cooling systems of buildings where the pressure difference is relatively small
- For a fixed surface area, the circular tube gives the most heat transfer for the least pressure drop, which explains the overwhelming popularity of circular tubes in heat transfer equipment
- Frictional heating must be considered for flows that involve highly viscous fluids with large velocity gradients
- In most practical applications, the flow of a fluid through a pipe or duct can be approximated to be *one-dimensional,* and thus the <u>properties can be assumed to vary only in direction of flow</u>

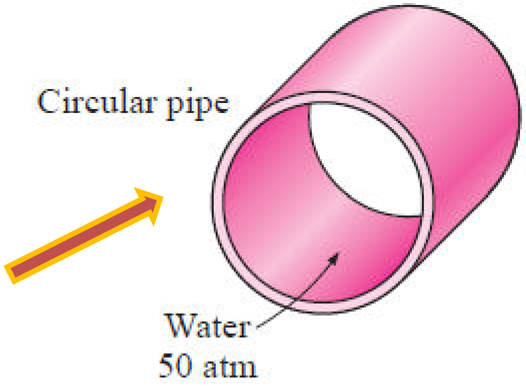


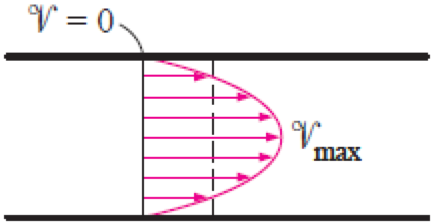


Mean velocity in a circular tube

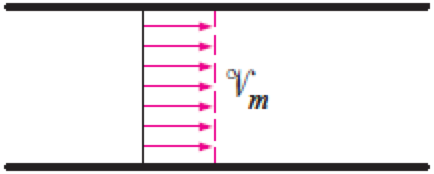
- In external flow, the free-stream velocity served as a convenient reference velocity for use in the evaluation of the Reynolds number and the friction coefficient

- In internal flow , the fluid velocity in a tube changes from zero at the surface because of the no-slip condition, to a maximum at the tube center. Therefore, it is convenient to represent **average** or **mean velocity** $\mathcal{V}_{m,.}$ For incompressible flow $\mathcal{V}_m$ is constant when cross sectional area is constant





(*a*) Actual



(*b*) Idealized

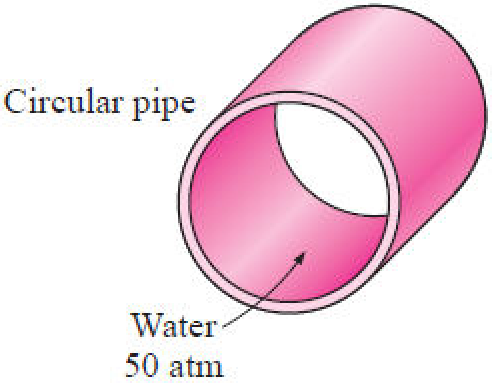
$$\dot{m} = \rho \mathcal{V}_m A_c = \int_{A_c} \rho \mathcal{V}(r, x) dA_c$$

where $\dot{m}$ is the mass flow rate, $\rho$is the density, A_c is the cross sectional area, and $\mathcal{V}$ (r, x) is the velocity profile. Then the mean velocity for incompressible flow in a circular tube of radius R can be expressed as:

$$\mathcal{V}_m = \frac{\int_{A_c} \rho \mathcal{V}(r, x) dA_c}{\rho A_c} = \frac{\int_0^R \rho \mathcal{V}(r, x) 2\pi r dr}{\rho \pi R^2} = \frac{2}{R^2} \int_0^R \mathcal{V}(r, x) r dr$$

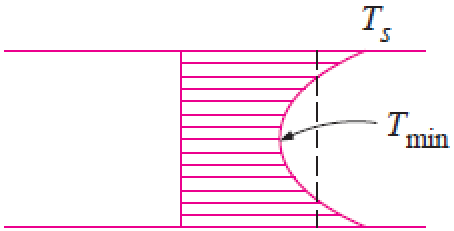
Mean temperature in a circular tube

- When a fluid is heated or cooled as it flows through a tube, the temperature of the fluid at any cross section changes from T_s at the surface of the wall to some maximum (or minimum in the case of heating) at the tube center. Therefore it is convenient **average** or **mean temperature** T_m that remains uniform at a cross section

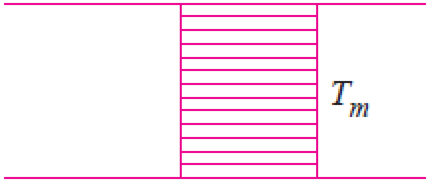


- As per conservation of energy principle, energy transported by the fluid through a cross section in actual flow must be equal to the energy that would be transported through the same cross section if the fluid were at a constant temperature T_m

$$\dot{E}_{\text{fluid}} = \dot{m} C_p T_m = \int_{\dot{m}} C_p T \, \delta\dot{m} = \int_{A_c} \rho C_p T \mathcal{V} \, dA_c$$



(*a*) Actual



(*b*) Idealized

where C_p is the specific heat of the fluid. Then the mean temperature of the fluid with constant density and specific heat flowing in circular pipe of radius R is:

$$T_m = \frac{\int_{\dot{m}} C_p T \, \delta\dot{m}}{\dot{m} C_p} = \frac{\int_0^R C_p T (\rho \mathcal{V} 2\pi r dr)}{\rho \mathcal{V}_m (\pi R^2) C_p} = \frac{2}{\mathcal{V}_m R^2} \int_0^R T(r, x) \, \mathcal{V}(r, x) \, r dr$$

Laminar and turbulent flow in the tubes

- Most pipe flows encountered in practice are turbulent. Laminar flow is encountered when highly viscous fluids such as oils flow in small diameter tubes or narrow passages

In Re calculation, $\mathcal{V}_m$ *is the mean fluid velocity, D is the diameter of the tube, and* $\nu = \mu/\rho$ *is the kinematic viscosity of the fluid*

- For flow through noncircular tubes, the Reynolds number as well as the Nusselt number and the friction factor are based on the **hydraulic diameter** D_h defined as:

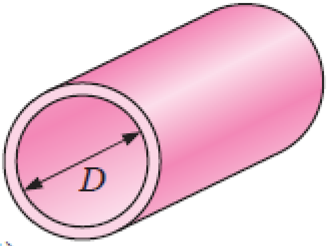
$$D_h = \frac{4A_c}{p}$$

where A_c is the cross sectional area of the tube and p is its perimeter

- Transition from laminar to turbulent flow also depends on the degree of disturbance of the flow by *surface roughness, pipe vibrations,* and the *fluctuations in the flow*

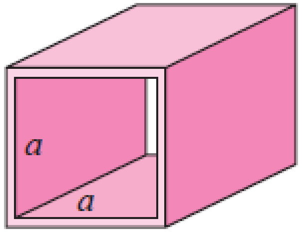
$$\mathrm{Re} = \frac{\rho \mathcal{V}_m D}{\mu} = \frac{\mathcal{V}_m D}{\nu}$$

Circular tube:



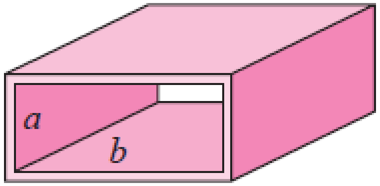
$$D_h = \frac{4(\pi D^2/4)}{\pi D} = D$$

Square duct:



$$D_h = \frac{4a^2}{4a} = a$$

Rectangular duct:

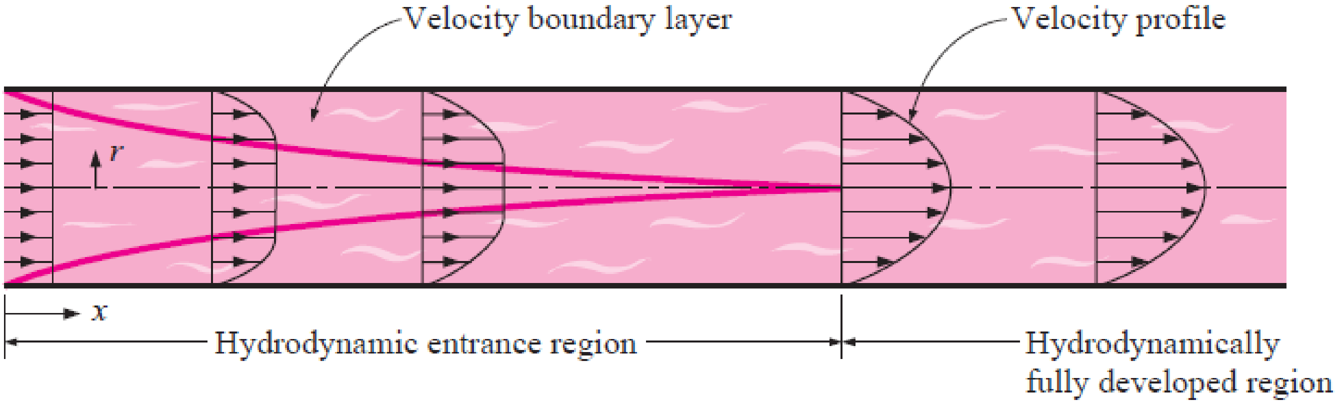


$$D_h = \frac{4ab}{2(a+b)} = \frac{2ab}{a+b}$$

$\mathrm{Re} < 2300$	laminar flow
$2300 \le \mathrm{Re} \le 10{,}000$	transitional flow
$\mathrm{Re} > 10{,}000$	turbulent flow

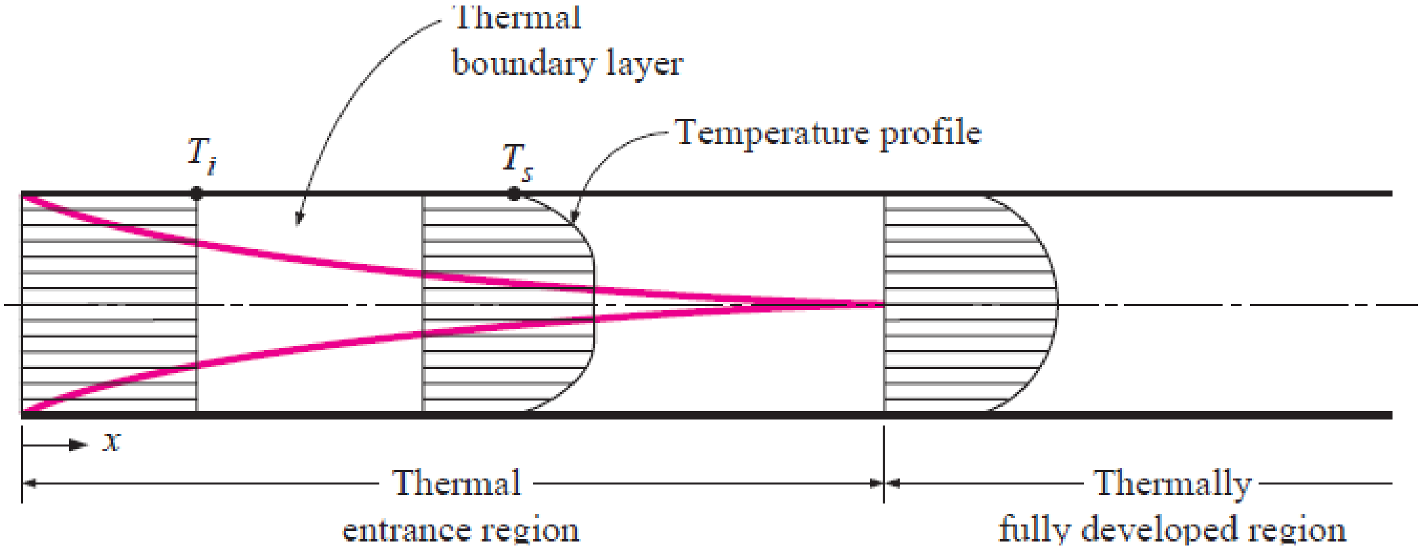
Hydrodynamic entrance region

- For a fluid entering circular tube at uniform velocity, the fluid particles in the layer in contact with the surface of the tube will come to a complete stop. Subsequently adjacent layers will slow down as a result of friction
- To make up for this velocity reduction, the velocity of the fluid at the midsection of the tube will have to increase to keep the mass flow rate through the tube constant. As a result, a *velocity boundary layer* develops along the tube
- The thickness of this boundary layer increases in the flow direction until the boundary layer reaches the tube center and thus fills the entire tube
- The region from the tube inlet to the point at which the boundary layer merges at the centerline is called the **hydrodynamic entrance region,** and the length of this region is called the **hydrodynamic entry length** L_h
- The region beyond the entrance region in which the velocity profile is fully developed and remains unchanged is called the **hydrodynamically fully developed region.** Here the velocity profile is parabolic in laminar flow and flatter in turbulent flow



Thermal entrance region

- Consider a fluid at a uniform temperature entering a circular tube whose surface is maintained at a different temperature. The fluid particles in the layer in contact with the surface of the tube will assume the surface temperature
- Due to convective heat transfer in the tube, thickness of thermal boundary layer formed also increases in the flow direction until the boundary layer reaches the tube center and thus fills the entire tube
- The region of flow over which the thermal boundary layer develops and reaches the tube center is called the **thermal entrance region,** and the length of this region is called the **thermal entry length** L_t
- The region beyond the thermal entrance region in which the dimensionless temperature profile expressed as $(T_s - T)/(T_s - T_m)$ remains unchanged is called the **thermally fully developed region**



Characteristics of hydrodynamically and thermally fully developed regions

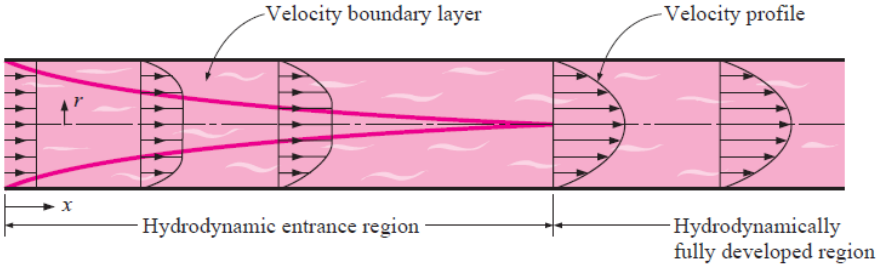
Hydrodynamically fully developed: $$\frac{\partial \mathcal{V}(r,x)}{\partial x} = 0 \longrightarrow \mathcal{V} = \mathcal{V}(r)$$

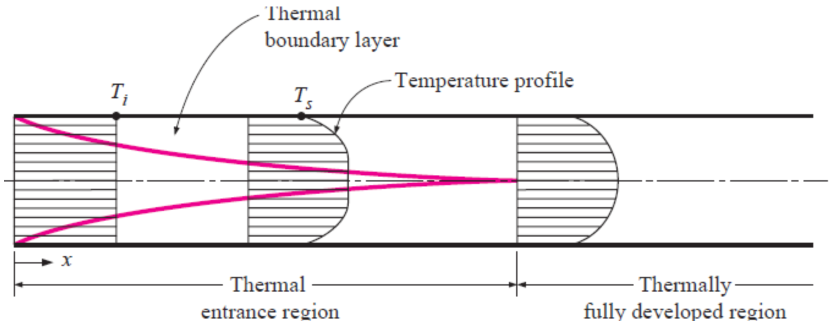
Thermally fully developed: $$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0$$

- The friction factor is related to the shear stress at the surface, which is related to the slope of the velocity profile at the surface. Since the velocity profile remains unchanged in fully developed region, friction factor remains unchanged too. A similar argument can be given for heat transfer coefficient in fully developed region
- The derivate $(T_s - T)/(T_s - T_m)$ with respect to r must also be independent of x

$$\frac{\partial}{\partial r}\left(\frac{T_s - T}{T_s - T_m}\right)\bigg|_{r=R} = \frac{-(\partial T/\partial r)|_{r=R}}{T_s - T_m} \neq f(x) \Rightarrow \dot{q}_s = h_x(T_s - T_m) = k\frac{\partial T}{\partial r}\bigg|_{r=R} \longrightarrow h_x = \frac{k(\partial T/\partial r)|_{r=R}}{T_s - T_m}$$

<u>Both *the friction and convection coefficients remain constant in the fully developed region of a tube*</u>





Characteristics of hydrodynamically and thermally fully developed regions

Hydrodynamically fully developed:

$$\frac{\partial \mathcal{V}(r, x)}{\partial x} = 0 \longrightarrow \mathcal{V} = \mathcal{V}(r)$$

Thermally fully developed:

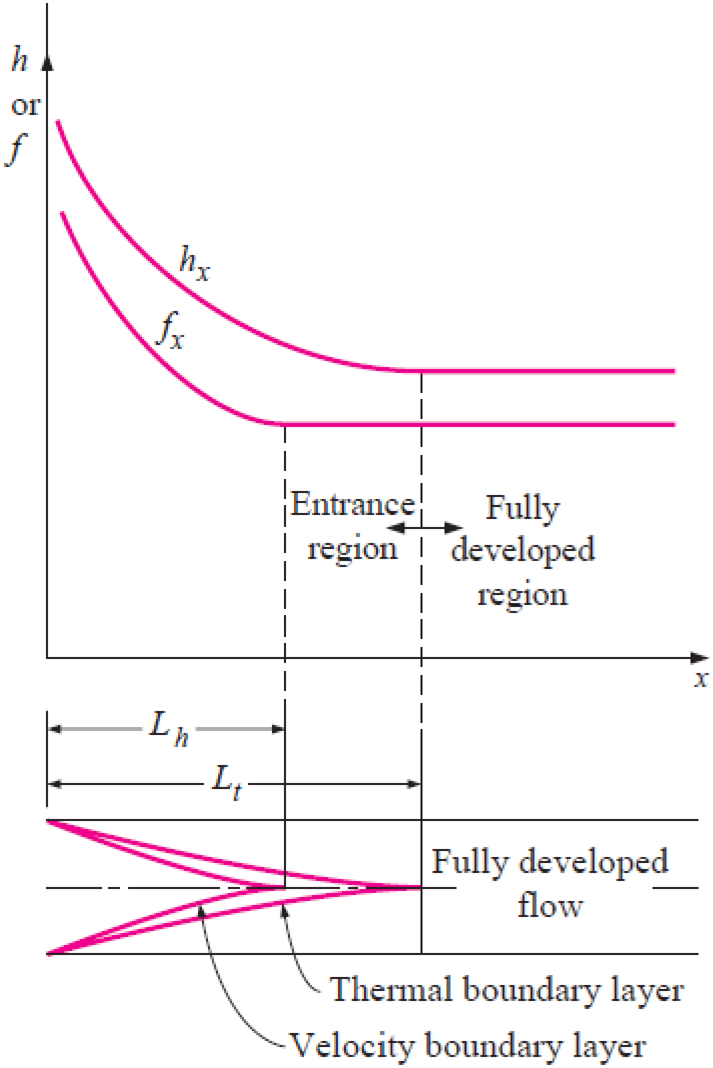
$$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r, x)}{T_s(x) - T_m(x)}\right] = 0$$

- Unlike the velocity profile, the temperature profile can be different at different cross sections of the tube in the developed region, and it usually is. However, the dimensionless temperature profile defined above remains unchanged in the thermally developed region when the temperature or heat flux at the tube surface remains constant

- For fluids with Pr ~ 1, such as gases, the two boundary layers essentially coincide with each other. For fluids with Pr >> 1, such as oils, the velocity boundary layer outgrows the thermal boundary layer. Hence hydrodynamic entry length is smaller than thermal entry length

Characteristics of hydrodynamically and thermally fully developed regions

- Consider a fluid that is being heated (or cooled) in a tube as it flows through it. The friction factor and the heat transfer coefficient are *highest* at the tube inlet where the thickness of the boundary layers is zero, and decrease gradually to the fully developed values
- Pressure drop and heat flux are *higher* in the entrance regions of a tube, and the effect of the entrance region is always to *enhance* the average friction and heat transfer coefficients for the entire tube
- This enhancement effect can be significant for short tubes but negligible for long ones



Magnitude of hydrodynamic and thermal entry lengths

In *laminar flow,* the hydrodynamic and thermal entry lengths are approximately:

$$L_{h,\ \mathrm{laminar}} \approx 0.05\ \mathrm{Re}\ D$$
$$L_{t,\ \mathrm{laminar}} \approx 0.05\ \mathrm{Re}\ \mathrm{Pr}\ D = \mathrm{Pr}\ L_{h,\ \mathrm{laminar}}$$

In the limiting case of Re = 2300, the hydrodynamic entry length is 115*D*

- In *turbulent flow,* the intense mixing during random fluctuations usually overshadows the effects of momentum and heat diffusion, and therefore the hydrodynamic and thermal entry lengths are of about the same size and independent of the Prandtl number.
- Also, *the friction factor and the heat transfer coefficient remain constant in fully developed laminar or turbulent flow* since the velocity and normalized temperature profiles do not vary in the flow direction

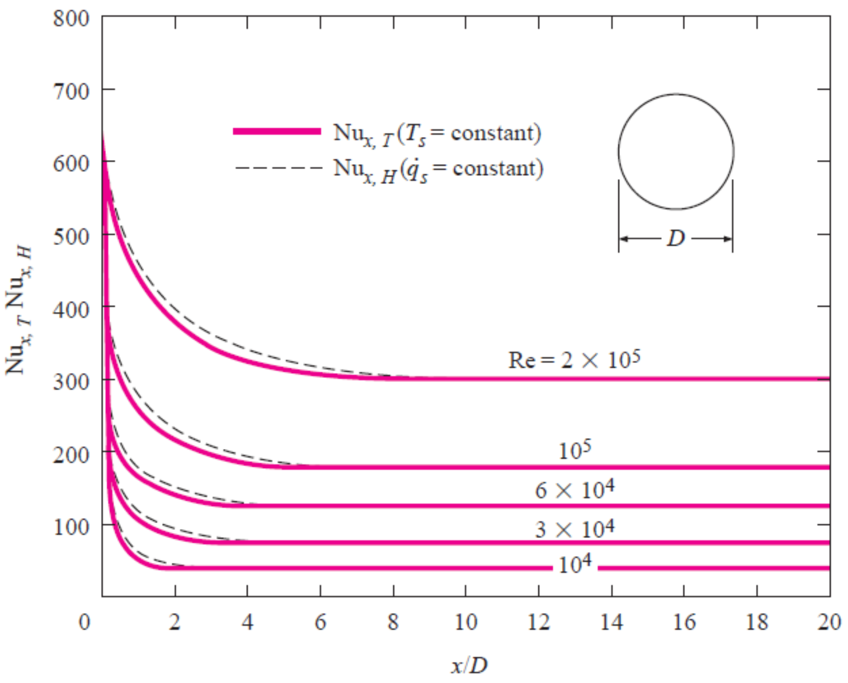
$$L_{h,\ \mathrm{turbulent}} = 1.359\ \mathrm{Re}^{1/4}$$

In practice, it is generally agreed that the entrance effects are confined to be:

$$L_{h,\ \mathrm{turbulent}} \approx L_{t,\ \mathrm{turbulent}} \approx 10D$$

Variation of Nusselt number in a turbulent flow

- The variation of local Nusselt number along a tube in turbulent flow for both uniform surface temperature and uniform surface heat flux is given below for the range of Reynolds numbers encountered in heat transfer equipment
- The Nusselt numbers and thus the convection heat transfer coefficients are much higher in the entrance region
- The Nusselt number reaches a constant value for $x > 10D$
- The Nusselt numbers for the uniform surface temperature and uniform surface heat flux conditions are identical in the fully developed regions, and nearly identical in the entrance regions. Therefore, Nusselt number is insensitive to the type of thermal boundary condition



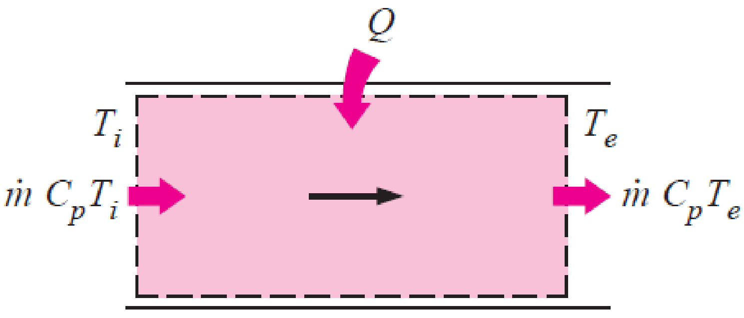
General thermal analysis of heat transfer to a fluid flowing in a tube

- The conservation of energy equation for the steady flow of a fluid in a tube is:

$$\dot{Q} = \dot{m} C_p (T_e - T_i)$$

where T_i and T_e are the mean fluid temperatures at the inlet and exit of the tube, respectively, and $\dot{Q}$ is the rate of heat transfer to or from the fluid. The temperature of a fluid flowing in a tube remains constant in the absence of any energy interactions through the wall of the tube

- The thermal conditions at the surface can usually be approximated with reasonable accuracy to be *constant surface temperature* (T_s = constant) or *constant surface heat flux* q_s = constant
- Constant temperature condition is realized when a phase change process such as boiling or condensation occurs at the outer surface of a tube. The constant surface heat flux condition is realized when the tube is subjected to radiation or electric resistance heating uniformly from all directions



Energy balance:

$$\dot{Q} = \dot{m} C_p (T_e - T_i)$$

Constant Surface Heat Flux ($\dot{q}_s = \text{constant}$)

In the case of q_s= constant, the rate of heat transfer:

$$\dot{Q} = \dot{q}_s A_s = \dot{m} C_p (T_e - T_i)$$

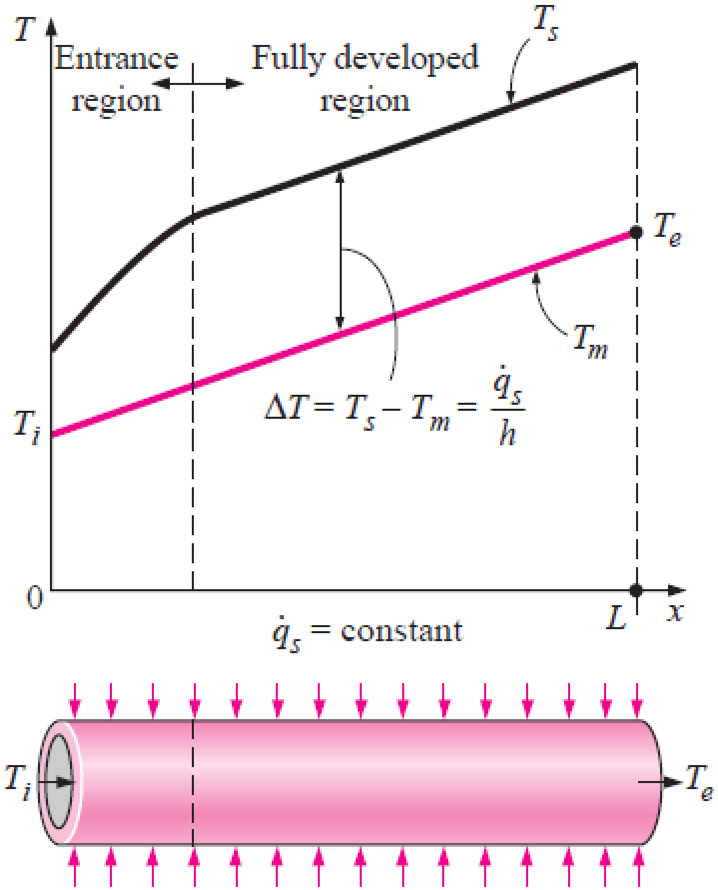
The mean fluid temperature at the tube exit is:

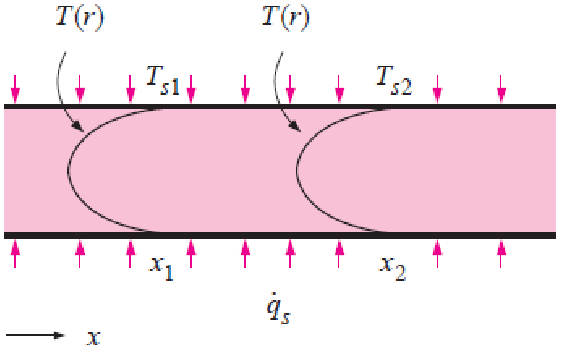
$$T_e = T_i + \frac{\dot{q}_s A_s}{\dot{m} C_p}$$

Mean fluid temperature increases *linearly* in the flow direction in the case of constant surface heat flux, since the surface area increases linearly in the flow direction

$$\dot{q}_s = h(T_s - T_m) \longrightarrow T_s = T_m + \frac{\dot{q}_s}{h}$$

In the fully developed region, the surface temperature T_s will also increase linearly in the flow direction since h is constant and thus $T_s - T_m$ = constant



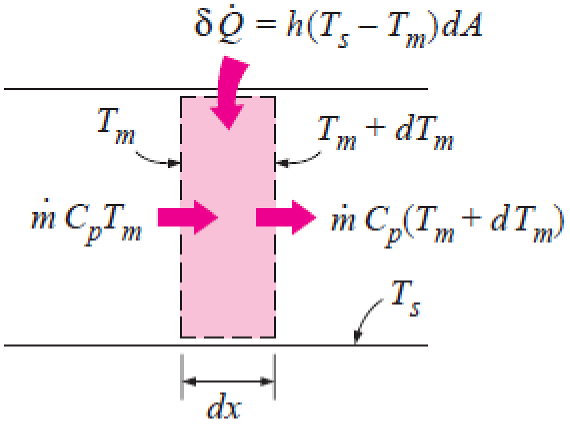


Constant Surface Heat Flux ($\dot{q}_s$ = constant)

The slope of the mean fluid temperature T_m on a *T-x* diagram can be determined by applying the steady-flow energy balance to a tube slice of thickness *dx*

$$\dot{m}\,C_p\,dT_m = \dot{q}_s(p\,dx) \longrightarrow \frac{dT_m}{dx} = \frac{\dot{q}_s p}{\dot{m}C_p} = \text{constant}$$

$$\frac{dT_m}{dx} = \frac{dT_s}{dx}$$



Also, the requirement that the dimensionless temperature profile remains unchanged in the fully developed region:

$$\frac{\partial}{\partial x}\left(\frac{T_s - T}{T_s - T_m}\right) = 0 \longrightarrow \frac{1}{T_s - T_m}\left(\frac{\partial T_s}{\partial x} - \frac{\partial T}{\partial x}\right) = 0 \longrightarrow \frac{\partial T}{\partial x} = \frac{dT_s}{dx}$$

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{\dot{q}_s p}{\dot{m}C_p} = \text{constant}$$

For a developed flow in a tube subjected to constant surface heat flux, the temperature gradient is independent of x

Hence for a circular tube:

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{2\dot{q}_s}{\rho \mathcal{V}_m C_p R} = \text{constant}$$

Constant Surface Temperature (T_s = constant)

From Newton's law of cooling, the rate of heat transfer to or from a fluid flowing in a tube:

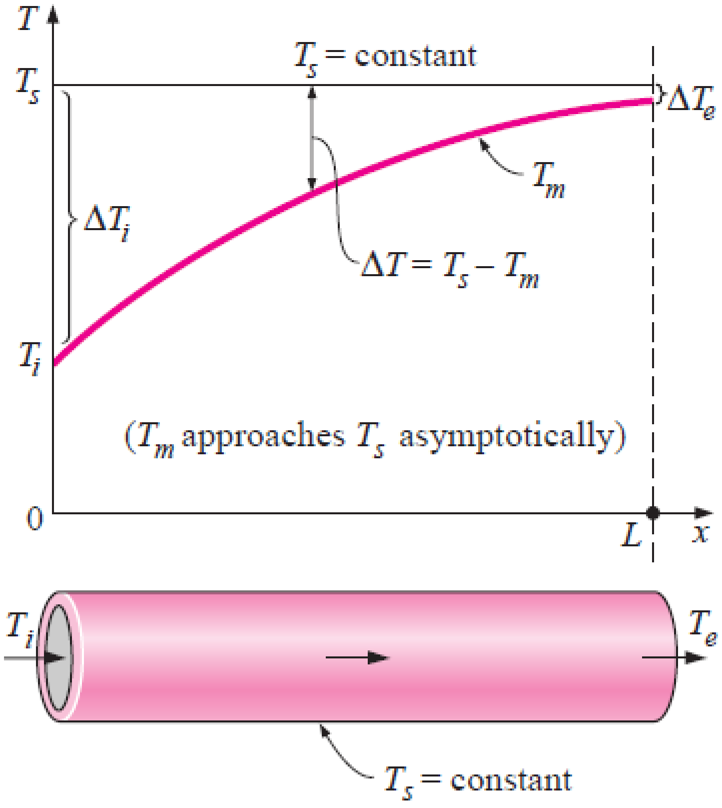
$$\dot{Q} = hA_s\Delta T_{\text{ave}} = hA_s(T_s - T_m)_{\text{ave}}$$

where *h* is the average convection heat transfer coefficient, $A_s = \pi DL$ for a circular pipe of length *L*), and T_{ave} is some appropriate *average* temperature difference between the fluid and the surface.

T_{ave} can be approximated by the **arithmetic mean temperature difference T_{am}**

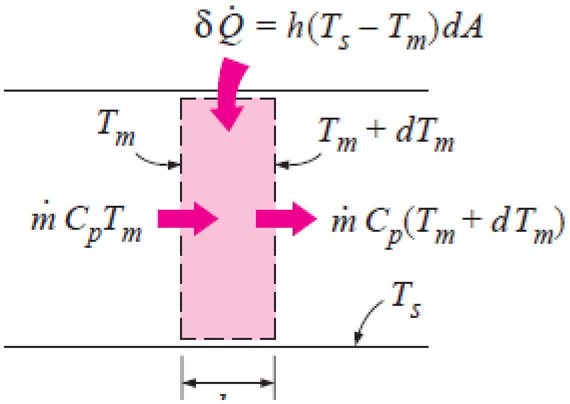
$$\Delta T_{\text{ave}} \approx \Delta T_{\text{am}} = \frac{\Delta T_i + \Delta T_e}{2} = \frac{(T_s - T_i) + (T_s - T_e)}{2} = T_s - \frac{T_i + T_e}{2}$$
$$= T_s - T_b$$

where $T_b = (T_i + T_e)/2$ is the *bulk mean fluid temperature,* which is the *arithmetic average* of the mean fluid temperatures at the inlet and the exit of the tube.



Constant Surface Temperature (T_s = constant)

Consider the heating of a fluid in a tube of constant cross section whose inner surface is maintained at a constant temperature of T_s. Mean temperature of the fluid T_m will increase in the flow direction as a result of heat transfer. The energy balance on a differential control volume shown in Fig:



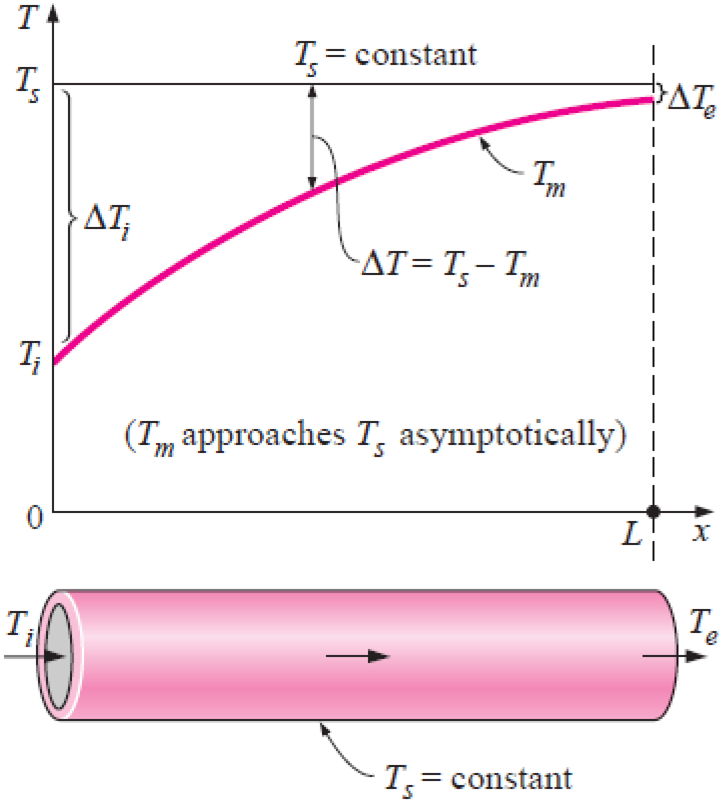
$$\dot{m}\, C_p\, dT_m = h(T_s - T_m)dA_s$$

Since differential surface area is $dA_s = pdx$, where p is the perimeter of the tube, and that $dT_m = -d(T_s - T_m)$, since T_s is constant, the relation above can be rearranged as

$$\frac{d(T_s - T_m)}{T_s - T_m} = -\frac{hp}{\dot{m}C_p}dx$$

Integrating from $x = 0$ (tube inlet where $T_m = T_i$) to $x = L$ (tube exit where $T_m = T_e$)

$$\ln\frac{T_s - T_e}{T_s - T_i} = -\frac{hA_s}{\dot{m}C_p}$$

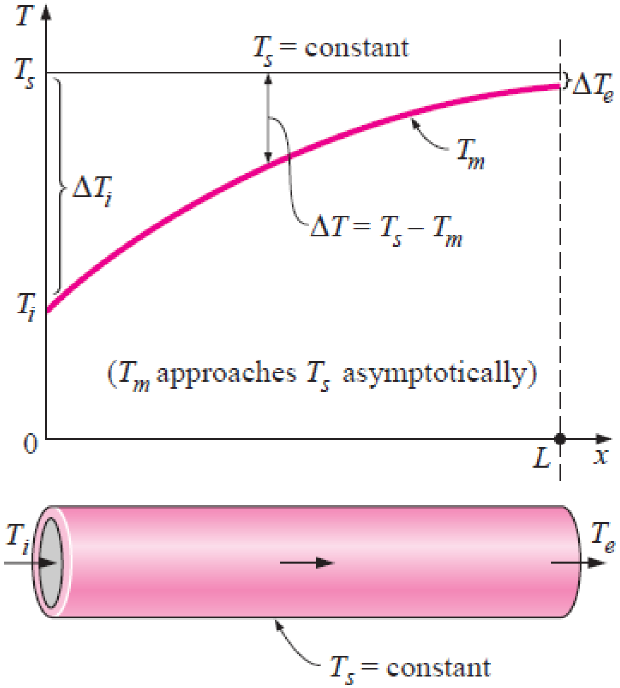


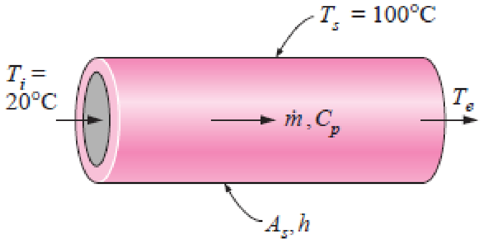
Constant Surface Temperature (T_s = constant)

- $A_s = pL$ is the surface area of the tube and h is the constant *average* convection heat transfer coefficient:

$$T_e = T_s - (T_s - T_i)\exp(-hA_s/\dot{m}C_p)$$

- This relation can also be used to determine the mean fluid temperature $T_m(x)$ at any x by replacing $A_s = pL$ by px
- Temperature difference between the fluid and the surface *decays exponentially* in the flow direction, and the rate of decay depends on the magnitude of the exponent $hA_x/\dot{m}\cdot C_p$, as shown in Fig.
- $hA_x/\dot{m}\cdot C_p$ is a dimensionless parameter and is termed as NTU (number of transfer units) and is a measure of effectiveness of heat transfer systems
- NTU > 5 indicates the exit temperature of the fluid is almost equal to surface temperature





NTU = $hA_s/\dot{m}C_p$	T_e, °C
0.01	20.8
0.05	23.9
0.10	27.6
0.50	51.5
1.00	70.6
5.00	99.5
10.00	100.0

Constant Surface Temperature (T_s = constant)

$$\dot{m}C_p = -\frac{hA_s}{\ln[(T_s - T_e)/(T_s - T_i)]}$$

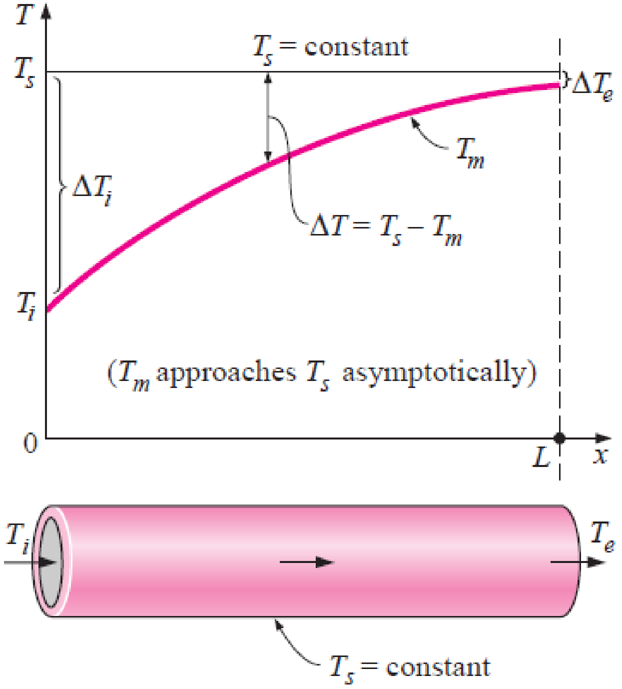
But it is known that:

$$\dot{Q} = \dot{q}_s A_s = \dot{m}C_p(T_e - T_i)$$

Therefore heat rate in constant surface temperature condition can be expressed as:

$$\dot{Q} = hA_s\Delta T_{\ln}$$

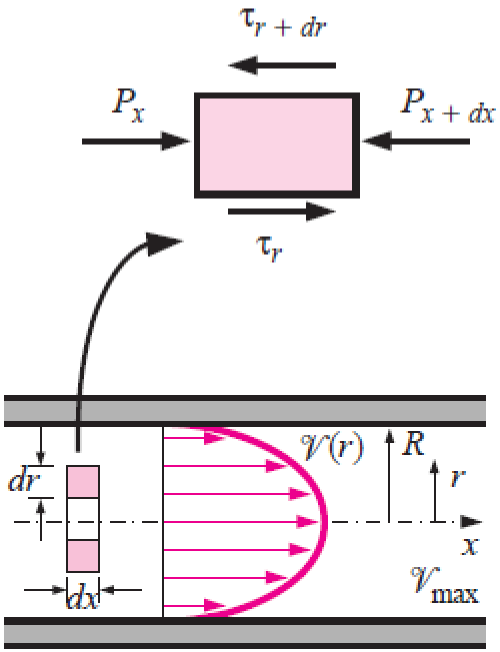
$$\Delta T_{\ln} = \frac{T_i - T_e}{\ln[(T_s - T_e)/(T_s - T_i)]} = \frac{\Delta T_e - \Delta T_i}{\ln(\Delta T_e/\Delta T_i)}$$



$\Delta T_{\ln}$ is termed as logarithmic mean temperature difference and is obtained by tracing actual temperature profile of the fluid along the tube and is the exact representation of average temperature difference between surface and the fluid at inlet

Laminar flow in the tubes

- Consider steady laminar flow of an incompressible fluid with constant properties in the fully developed region of a straight circular tube
- We obtain the momentum and energy equations by applying momentum and energy balances to a differential volume element, and acquire the velocity and temperature profiles by solving them
- Velocity and temperature profiles will be further used to obtain relations for the friction factor and the Nusselt number

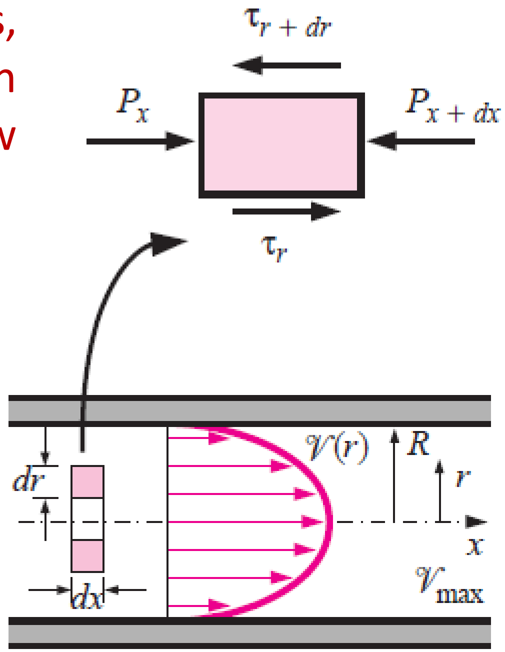


Assumptions in the derivations:

- *In fully developed laminar flow, each fluid particle moves at a constant axial velocity along a streamline and the velocity profile V(r) remains unchanged in the flow direction*
- *There is no motion in the radial direction, and thus the velocity component v in the direction normal to flow is everywhere zero. There is no acceleration since the flow is steady*

Laminar flow in the tubes

- Consider a ring-shaped differential volume element of radius *r*, thickness *dr* and length *dx* oriented coaxially with the tube, as shown in Fig.



- The volume element involves only pressure and viscous effects, and thus the pressure and shear forces must balance each other. A force balance on the volume element in the flow direction gives :

$$(2\pi r dr P)_x - (2\pi r dr P)_{x+dx} + (2\pi r dx \tau)_r - (2\pi r dx \tau)_{r+dr} = 0$$

Dividing by 2π*drdx* and rearranging:

$$r\frac{P_{x+dx} - Px}{dx} + \frac{(r\tau)_{r+dr} - (r\tau)_r}{dr} = 0$$

Taking the limit as *dr, dx* → 0 gives

$$r\frac{dP}{dx} + \frac{d(r\tau)}{dr} = 0$$

Substituting τ = −μ(dV/dr) and rearranging the desired equation

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{d\mathcal{V}}{dr}\right) = \frac{dP}{dx}$$

Contd-----

Laminar flow in the tubes

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{d\mathcal{V}}{dr}\right)=\frac{dP}{dx}$$

The above equality must hold for any value of *r* and *x,* and an equality of the form *f*(*r*) = *g*(*x*) can happen only if both *f*(*r*) and *g*(*x*) are constants. Thus we conclude that *dP*/*dx* is constant

The above equation can be solved by rearranging and integrating it twice to give:

$$V(r) = \frac{r^2}{4\mu}(\frac{dP}{dx})+C_1 lnr+C_2$$

The velocity profile V(*r*) is obtained by applying the boundary conditions ∂V/∂r = 0 at *r* = 0 (because of symmetry about the centerline) and V= 0 at *r* = *R* (the no-slip condition at the tube surface)

$$\mathcal{V}(r)=-\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1-\frac{r^2}{R^2}\right)$$

Therefore, the velocity profile in fully developed laminar flow in a tube is *parabolic* with a maximum at the centerline and minimum at the tube surface

Contd-----

Laminar flow in the tubes

The mean velocity is determined from its definition:

$$\mathcal{V}_m = \frac{2}{R^2}\int_0^R \mathcal{V} r dr = \frac{-2}{R^2}\int_0^R \frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right) r dr = -\frac{R^2}{8\mu}\left(\frac{dP}{dx}\right)$$

The mathematical relation between V(r) and V_m can be established by:

$$\mathcal{V}(r) = 2\mathcal{V}_m\left(1 - \frac{r^2}{R^2}\right)$$

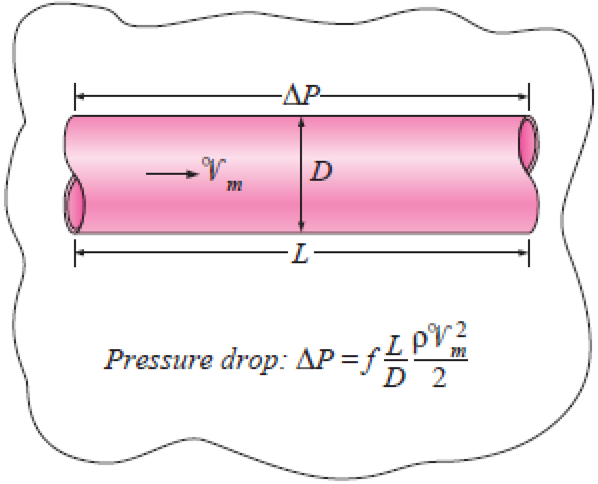
The maximum velocity occurs at the centerline, and is determined from the above equation by substituting *r* = 0

$$\mathcal{V}_{\max} = 2\mathcal{V}_m$$

Pressure Drop:

We note that dP/dx constant, and integrate it from x = 0 where the pressure is P_1 to x = L where the pressure is P_2

$$\frac{dP}{dx} = \frac{P_2 - P_1}{L} = -\frac{\Delta P}{L}$$



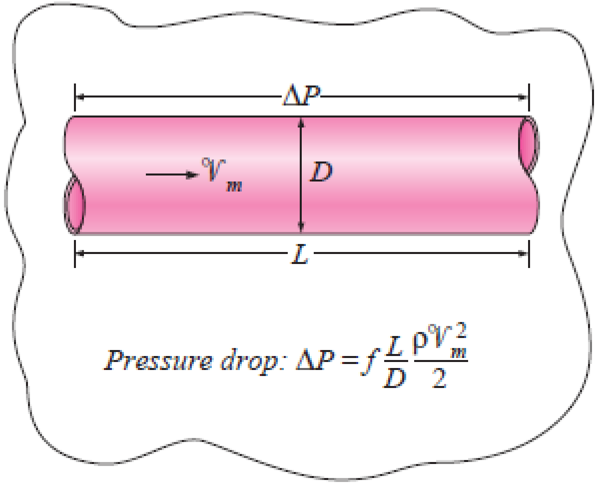
Pressure drop: $\Delta P = f\frac{L}{D}\frac{\rho\mathcal{V}_m^2}{2}$

Laminar flow in the tubes

Pressure Drop:

$$\frac{dP}{dx} = \frac{P_2 - P_1}{L} = -\frac{\Delta P}{L}$$

But $V_m = -\frac{R^2}{8\mu}\left(\frac{dP}{dx}\right)$



Pressure drop: $\Delta P = f\frac{L}{D}\frac{\rho\mathcal{V}_m^2}{2}$

Substituting $-\frac{\Delta P}{L}$ for $\frac{dP}{dx}$ in the above equation:

$$\Delta P = \frac{8\mu L\,\mathcal{V}_m}{R^2} = \frac{32\mu L\,\mathcal{V}_m}{D^2}$$

In practice, it is found convenient to express the pressure drop for all types of internal flows (laminar or turbulent flows, circular or noncircular tubes, smooth or rough surfaces) by assuming: (1)the flow section is horizontal so that there are no hydrostatic or gravity effects, (2) the flow section does not involve any work devices such as a pump or a turbine since they change the fluid pressure, and (3) the cross sectional area of the flow section is constant and thus the mean flow velocity is constant

$$\Delta P = f\frac{L}{D}\frac{\mu\mathcal{V}_m^2}{2}$$

where the dimensionless quantity *f* is the **friction factor** also called Darcy friction factor

For a circular tube subjected to laminar flow, friction factor is a function of Re

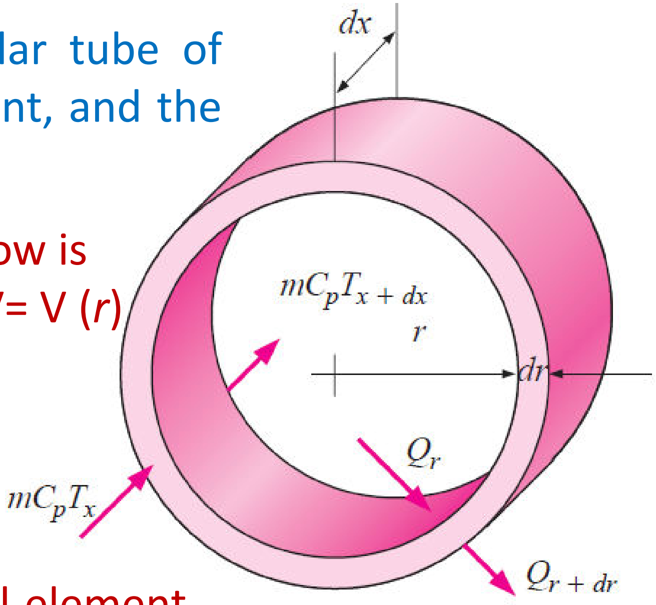
$$f = \frac{64\mu}{\rho D\mathcal{V}_m} = \frac{64}{\text{Re}}$$

Laminar flow in the tubes

Temperature Profile and the Nusselt Number

Assumptions in the derivation:

- Consider steady laminar flow of a fluid in a circular tube of radius *R*. The fluid properties ρ, *k*, and C_p are constant, and the work done by viscous stresses is negligible
- The fluid flows along the *x*-axis with velocity, V. The flow is fully developed so that is independent of *x* and thus V= V (*r*)
- Energy is transferred by mass in the *x*-direction, and by conduction in the *r*-direction (heat conduction in the *x*-direction is assumed to be negligible)



- The steady-flow energy balance for a cylindrical shell element of thickness *dr* and length *dx* can be expressed as:

$$\dot{m}C_pT_x - \dot{m}C_pT_{x+dx} + \dot{Q}_r - \dot{Q}_{r+dr} = 0$$

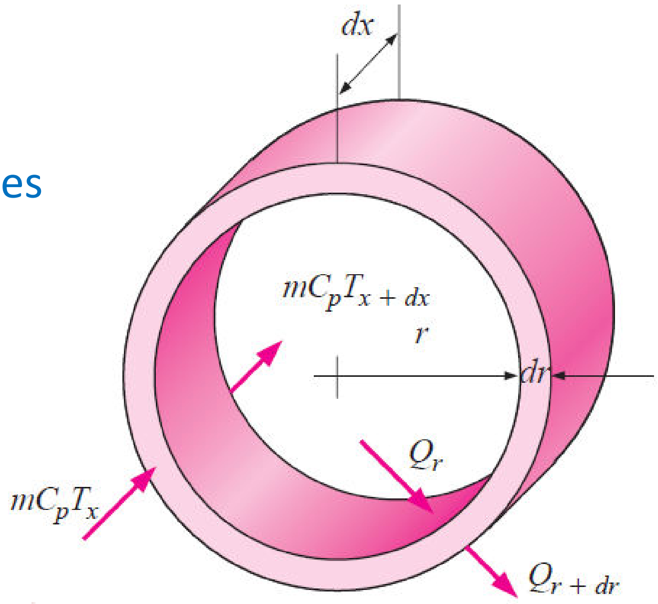
where $\dot{m}$ = ρVA$_c$= ρV(2πrdr)

Substituting and dividing by 2π*rdrdx and rearranging gives:*

$$\rho C_p \mathcal{V} \frac{T_{x+dx} - T_x}{dx} = -\frac{1}{2\pi r dx} \frac{\dot{Q}_{r+dr} - \dot{Q}_r}{dr}$$

Laminar flow in the tubes

Temperature Profile and the Nusselt Number



$$\rho C_p \mathcal{V} \frac{T_{x+dx} - T_x}{dx} = -\frac{1}{2\pi r dx} \frac{\dot{Q}_{r+dr} - \dot{Q}_r}{dr}$$

By taking limits on L.H.S and R.H.S w.r.t dx, dr → 0 gives

$$\mathcal{V} \frac{\partial T}{\partial x} = -\frac{1}{2\rho C_p \pi r dx} \frac{\partial \dot{Q}}{\partial r}$$

But conduction heat transfer $\dot{Q}$ is given by Fourier's law of heat conduction:

$$\frac{\partial \dot{Q}}{\partial r} = \frac{\partial}{\partial r}\left(-k 2\pi r dx \frac{\partial T}{\partial r}\right) = -2\pi k dx \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)$$

Substituting and using $\alpha = k/\rho C_p$ gives

$$\mathcal{V} \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{dr}\left(r \frac{\partial T}{\partial r}\right)$$

The rate of net energy transfer to the control volume by mass flow is equal to the net rate of heat conduction in the radial direction

Contd----

Laminar flow in the tubes

Temperature Profile and the Nusselt Number - Constant Surface Heat Flux condition

For fully developed flow in a circular pipe subjected to constant surface heat flux:

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{2\dot{q}_s}{\rho \mathcal{V}_m C_p R} = \text{constant}$$

Equation -i

The governing differential equation for temperature field in circular tube is:

$$\mathcal{V}\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{dr}\left(r\frac{\partial T}{\partial r}\right)$$

Equation -ii

But the velocity at a radius and mean velocity are related by:

$$\mathcal{V}(r) = 2\mathcal{V}_m\left(1 - \frac{r^2}{R^2}\right)$$

Equation -iii

Substituting Eqn.iii and Eq.i in Eqn.ii gives :

$$\frac{4\dot{q}_s}{kR}\left(1 - \frac{r^2}{R^2}\right) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

The above equation is a second order differential equation , the general solution is obtained by separating variables and integrating twice:

Contd----

Laminar flow in the tubes

Temperature Profile and the Nusselt Number - Constant Surface Heat Flux condition

By integrating twice :

$$\frac{4\dot{q}_s}{kR}\left(1-\frac{r^2}{R^2}\right)=\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

$$T(r) = \frac{\dot{q}_s}{kR}\left(r^2-\frac{r^4}{4R^2}\right)+C_1 \ln r+C_2$$

The above equation is subject to boundary conditions ,at r=0, $\frac{\partial T}{\partial r} = 0$ and at r=R, $T = T_s$

$$T = T_s-\frac{\dot{q}_s R}{k}\left(\frac{3}{4}-\frac{r^2}{R^2}+\frac{r^4}{4R^4}\right)$$

But we know the man temperature and mean velocity in a circular tube is given by:

$$T_m=\frac{\int_m C_p T\,\delta\dot{m}}{\dot{m}C_p}=\frac{\int_0^R C_p T(\rho\mathcal{V}2\pi r dr)}{\rho\mathcal{V}_m(\pi R^2)C_p}=\frac{2}{\mathcal{V}_m R^2}\int_0^R T(r,x)\,\mathcal{V}(r,x)\,r dr$$

$$\mathcal{V}(r)=2\mathcal{V}_m\left(1-\frac{r^2}{R^2}\right)$$

Substituting mean temperature and mean velocity in temperature field equation we get:

$$T_m = T_s-\frac{11}{24}\frac{\dot{q}_s R}{k} \Rightarrow h=\frac{24}{11}\frac{k}{R}=\frac{48}{11}\frac{k}{D}=4.36\frac{k}{D} \Rightarrow \text{Nu}=\frac{hD}{k}=4.36$$

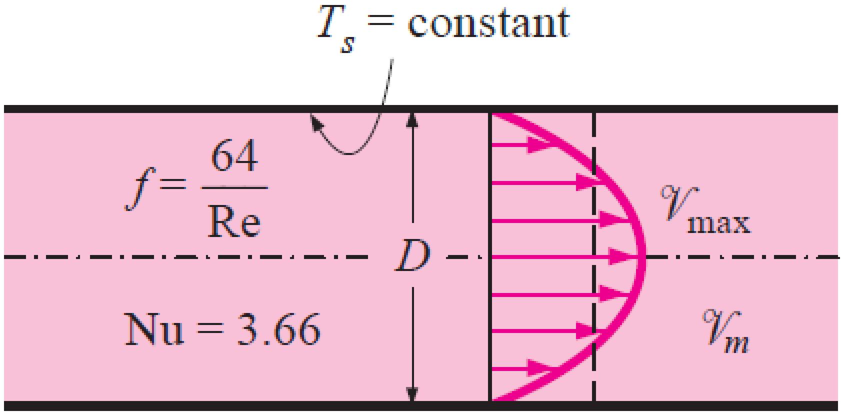
Laminar flow in the tubes

Temperature Profile and the Nusselt Number - Constant Surface temperature condition

- The solution procedure for constant surface temperature is more complex as it requires iterations, but the final Nusselt number relation obtained is equally simple as that of constant surface heat flux condition:

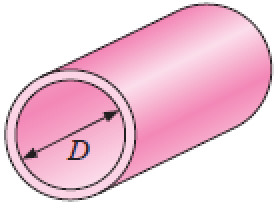
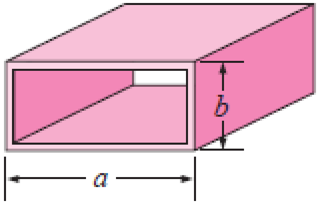
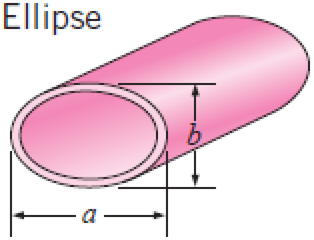
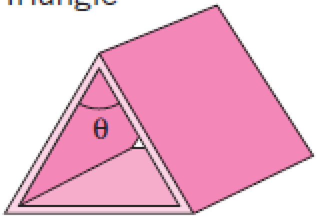
$$\mathrm{Nu} = \frac{hD}{k} = 3.66$$

- The thermal conductivity *k* for use in the Nu relations above should be evaluated at the bulk mean fluid temperature, which is the arithmetic average of the mean fluid temperatures at the inlet and the exit of the tube
- For laminar flow, the effect of *surface roughness* on the friction factor and the heat transfer coefficient is negligible



Fully developed
laminar flow

Laminar Flow in Noncircular Tubes

Tube Geometry	a/b or $\theta°$	*Nusselt Number*		Friction Factor f
		T_s = Const.	$\dot{q}_s$ = Const.	
Circle  	—	3.66	4.36	64.00/Re
Rectangle	a/b			
 	1	2.98	3.61	56.92/Re
	2	3.39	4.12	62.20/Re
	3	3.96	4.79	68.36/Re
	4	4.44	5.33	72.92/Re
	6	5.14	6.05	78.80/Re
	8	5.60	6.49	82.32/Re
	∞	7.54	8.24	96.00/Re
Ellipse	a/b			
 	1	3.66	4.36	64.00/Re
	2	3.74	4.56	67.28/Re
	4	3.79	4.88	72.96/Re
	8	3.72	5.09	76.60/Re
	16	3.65	5.18	78.16/Re
Triangle	θ			
 	10°	1.61	2.45	50.80/Re
	30°	2.26	2.91	52.28/Re
	60°	2.47	3.11	53.32/Re
	90°	2.34	2.98	52.60/Re
	120°	2.00	2.68	50.96/Re

Convection correlations for developing laminar flow in entrance region

- For a circular tube of length *L* subjected to constant surface temperature, the average Nusselt number for the *thermal entrance region is:*

$$\mathrm{Nu} = 3.66 + \frac{0.065\,(D/L)\,\mathrm{Re}\,\mathrm{Pr}}{1 + 0.04[(D/L)\,\mathrm{Re}\,\mathrm{Pr}]^{2/3}}$$

- Nusselt number is larger at the entrance region, and it approaches asymptotically to the fully developed value of 3.66 as $L \rightarrow \infty$

- When the difference between the surface and the fluid temperatures is large, it may be necessary to account for the variation of viscosity with temperature

- The average Nusselt number for developing laminar flow in a circular tube in that case can be determined as:

$$\mathrm{Nu} = 1.86\left(\frac{\mathrm{Re}\,\mathrm{Pr}\,D}{L}\right)^{1/3}\left(\frac{\mu_b}{\mu_s}\right)^{0.14}$$

- All properties are evaluated at the bulk mean fluid temperature, except for μ_s, which is evaluated at the surface temperature

Turbulent flow in the tubes

- Turbulent flow is commonly utilized in practice because of the higher heat transfer coefficients associated with it. Most correlations for the friction and heat transfer coefficients in turbulent flow are based on experimental studies because of the difficulty in dealing with turbulent flow theoretically

- For *smooth* tubes, the friction factor in turbulent flow can be determined from the explicit *first Petukhov equation:*

$$f = (0.790 \ln \mathrm{Re} - 1.64)^{-2} \qquad 10^4 < \mathrm{Re} < 10^6$$

- Once the friction factor is available, this equation can be used conveniently to evaluate the Nusselt number for both smooth and rough tubes

$$\mathrm{Nu} = 0.125 f \mathrm{Re}\mathrm{Pr}^{1/3}$$

- For fully developed turbulent flow in *smooth tubes,* a simple relation for the Nusselt number can be obtained by substituting the simple power law relation $f = 0.184\ \mathrm{Re}^{-0.2}$ for the friction factor in the above equation:

$$\mathrm{Nu} = 0.023\ \mathrm{Re}^{0.8}\ \mathrm{Pr}^{1/3} \qquad \begin{pmatrix} 0.7 \le \mathrm{Pr} \le 160 \\ \mathrm{Re} > 10{,}000 \end{pmatrix}$$

Contd----

Turbulent flow in the tubes

$$\mathrm{Nu} = 0.023\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{n}$$

where n = 0.4 for heating and 0.3 for cooling of the fluid flowing through the tube. This equation is known as the Dittus–Boelter equation

- The fluid properties are evaluated at the *bulk mean fluid temperature* $T_b = (T_i + T_e)/2$
- When the temperature difference between the fluid and the wall is very large, it may be necessary to use a correction factor to account for the different viscosities near the wall and at the tube center
- Gnielinski's equation should be preferred in calculations because it greatly eliminate error . Again properties should be evaluated at the bulk mean fluid temperature

$$\mathrm{Nu} = \frac{(f/8)(\mathrm{Re} - 1000)\,\mathrm{Pr}}{1 + 12.7(f/8)^{0.5}\,(\mathrm{Pr}^{2/3} - 1)} \qquad \left(\begin{array}{l} 0.5 \le \mathrm{Pr} \le 2000 \\ 3 \times 10^3 < \mathrm{Re} < 5 \times 10^6 \end{array}\right)$$

- The relations above are not very sensitive to the *thermal conditions* at the tube surfaces and can be used for both $\dot{T}_s$ = constant and $\dot{q}_s$ = constant cases

Turbulent flow in the tubes – Role of rough surfaces

- Any irregularity or roughness on the surface disturbs the laminar sublayer, and affects the flow. Therefore, unlike laminar flow, the friction factor and the convection coefficient in turbulent flow are strong functions of surface roughness

- The friction factor in fully developed turbulent flow depends on the Reynolds number and the *relative roughness ε/D and is given by Colebrook implicit equation:*

$$\frac{1}{\sqrt{f}} = -2.0 \log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\,\sqrt{f}}\right) \qquad \text{(turbulent flow)}$$

Standard sizes for Schedule 40 steel pipes

Nominal Size, in.	Actual Inside Diameter, in.
⅛	0.269
¼	0.364
⅜	0.493
½	0.622
¾	0.824
1	1.049
1½	1.610
2	2.067
2½	2.469
3	3.068
5	5.047
10	10.02

Relative Roughness, ε/L	Friction Factor, *f*
0.0*	0.0119
0.00001	0.0119
0.0001	0.0134
0.0005	0.0172
0.001	0.0199
0.005	0.0305
0.01	0.0380
0.05	0.0716

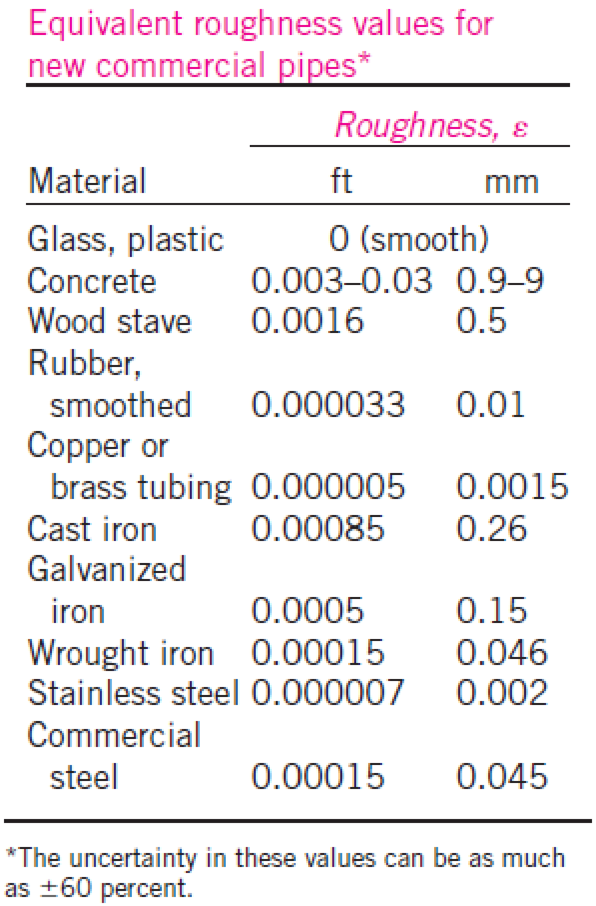
*Smooth surface. All values are for Re = 10^6,

Turbulent flow in the tubes – Role of rough surfaces

- Commercially available pipes differ from those used in the experiments in that the roughness of pipes in the market is not uniform, and it is difficult to give a precise description of it.
- The relative roughness of pipes may increase with use as a result of corrosion, scale buildup, and precipitation. As a result, the friction factor may increase by a factor of 5 to 10
- A simplified explicit equation for f is given by:

$$\frac{1}{\sqrt{f}} \approx -1.8 \log\left[\frac{6.9}{\text{Re}} + \left(\frac{\varepsilon/D}{3.7}\right)^{1.11}\right]$$

- In turbulent flow, wall roughness increases the heat transfer coefficient *h* by a factor of 2 or more



Equivalent roughness values for new commercial pipes*

	Roughness, ε	
Material	ft	mm
Glass, plastic	0 (smooth)	
Concrete	0.003–0.03	0.9–9
Wood stave	0.0016	0.5
Rubber, smoothed	0.000033	0.01
Copper or brass tubing	0.000005	0.0015
Cast iron	0.00085	0.26
Galvanized iron	0.0005	0.15
Wrought iron	0.00015	0.046
Stainless steel	0.000007	0.002
Commercial steel	0.00015	0.045

*The uncertainty in these values can be as much as ±60 percent.

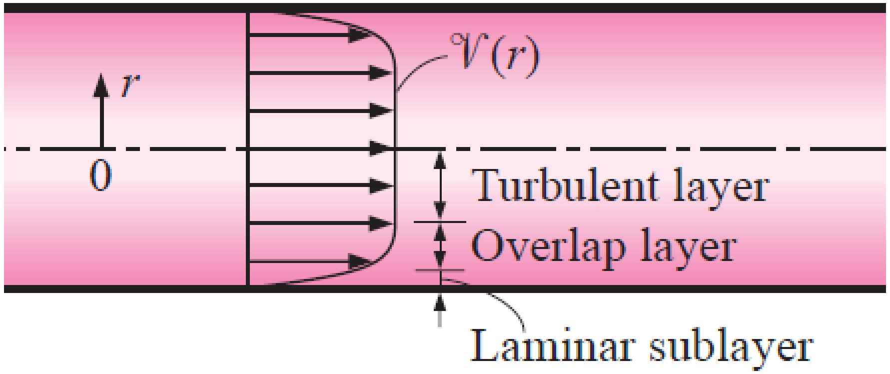
Turbulent Flow in Noncircular Tubes

- The velocity and temperature profiles in turbulent flow are nearly straight lines in the core region, and any significant velocity and temperature gradients occur in the viscous sublayer

- Despite the small thickness of laminar sublayer (usually much less than 1 percent of the pipe diameter), the characteristics of the flow in this layer are very important since they set the stage for flow in the rest of the pipe

- Consequently, the turbulent flow relations given above for circular tubes can also be used for noncircular tubes with reasonable accuracy by replacing the diameter *D* in the evaluation of the Reynolds number by the hydraulic diameter, D_h

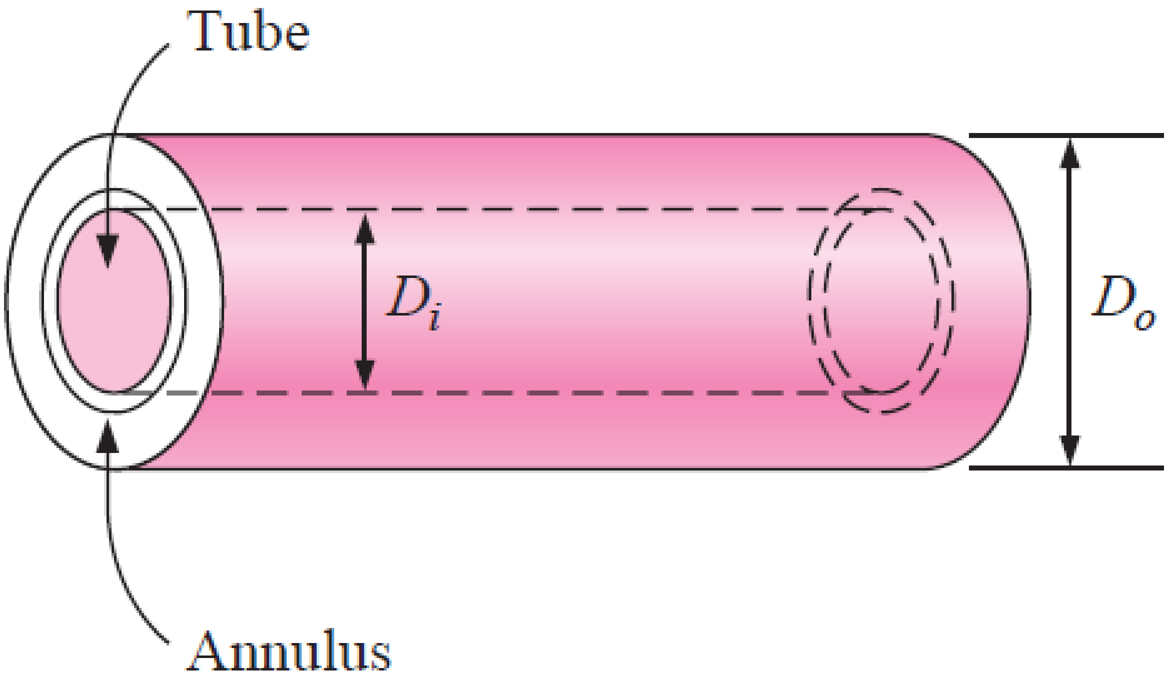
$$D_h = \frac{4A_c}{p}$$



Flow through Tube Annulus

- Some simple heat transfer equipment consist of two concentric tubes, and are called double-tube heat exchangers
- In these devices, one fluid flows through the tube while the other flows through the annular space. The governing differential equations for both flows are identical
- Consider a concentric annulus of inner diameter D_i and outer diameter D_o. The hydraulic diameter of annulus is:

$$D_h = \frac{4A_c}{p} = \frac{4\pi(D_o^2 - D_i^2)/4}{\pi(D_o + D_i)} = D_o - D_i$$



Flow through Tube Annulus

- Annular flow is associated with two Nusselt numbers—Nu_i on the inner tube surface and Nu_o on the outer tube surface—since it may involve heat transfer on both surfaces

$$\mathrm{Nu}_i = \frac{h_i D_h}{k} \qquad \text{and} \qquad \mathrm{Nu}_o = \frac{h_o D_h}{k}$$

- The Nusselt numbers for fully developed laminar flow with one surface isothermal and the other adiabatic are given as:

Nusselt number for fully developed laminar flow in an annulus with one surface isothermal and the other adiabatic

D_i/D_o	Nu_i	Nu_o
0	—	3.66
0.05	17.46	4.06
0.10	11.56	4.11
0.25	7.37	4.23
0.50	5.74	4.43
1.00	4.86	4.86

- For fully developed turbulent flow, the inner and outer convection coefficients are approximately equal to each other, and the tube annulus can be treated as a noncircular duct with a hydraulic diameter of $D_h = D_o - D_i$.
- The Nusselt number in this case can be determined from a suitable turbulent flow relation such as the Gnielinski's equation

$$\mathrm{Nu} = \frac{(f/8)(\mathrm{Re} - 1000)\,\mathrm{Pr}}{1 + 12.7(f/8)^{0.5}(\mathrm{Pr}^{2/3} - 1)} \qquad \begin{pmatrix} 0.5 \le \mathrm{Pr} \le 2000 \\ 3 \times 10^3 < \mathrm{Re} < 5 \times 10^6 \end{pmatrix}$$

- To improve the accuracy of Nusselt numbers obtained from these relations for annular flow, Petukhov and Roizen recommended correction factors as:

$$F_i = 0.86\left(\frac{D_i}{D_o}\right)^{-0.16} \qquad \text{(outer wall adiabatic)}$$

$$F_o = 0.86\left(\frac{D_i}{D_o}\right)^{-0.16} \qquad \text{(inner wall adiabatic)}$$

HEAT TRANSFER - 80329

MODULE - IV

NATURAL CONVECTION, BOILING AND CONDENSATION

MALLA REDDY ENGINEERING COLLEGE

MREC

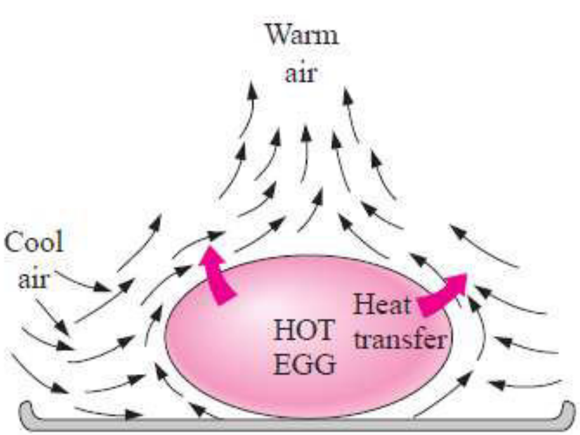
Natural Convection

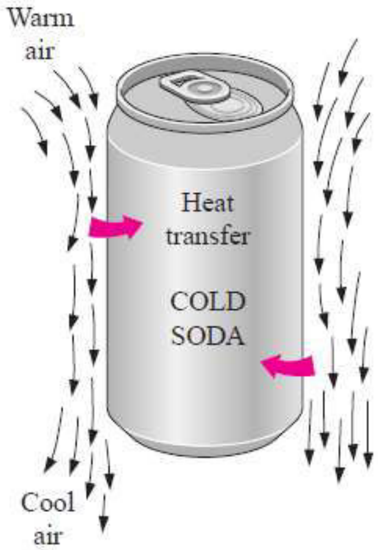
Part 1

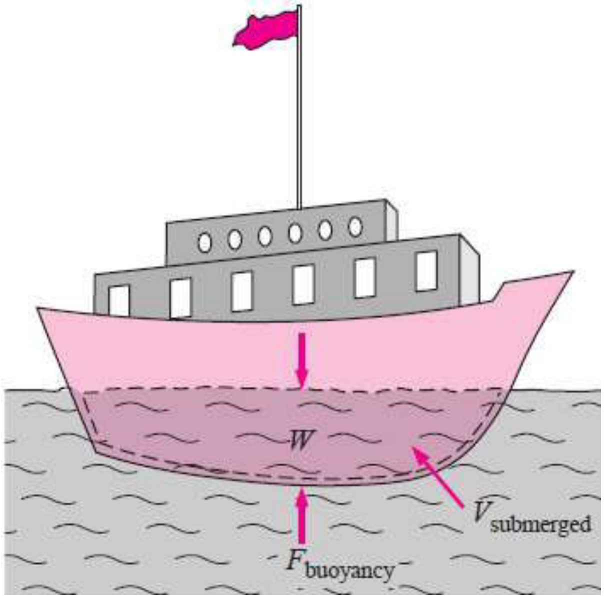
Dr. Satish Idury
Associate Professor

Natural Convection - Introduction

- The temperature of the air adjacent to the egg is higher, and thus its density is lower, since at constant pressure the density of a gas is inversely proportional to its temperature
- The motion that results from the continual replacement of the heated air in the vicinity of the egg by the cooler air nearby is called a **natural convection current**
- Natural convection is just as effective in the heating of cold surfaces in a warmer environment as it is in the cooling of hot surfaces in a cooler environment
- In a gravitational field, there is a net force that pushes upward a light fluid placed in a heavier fluid. This upward force is termed as buoyancy







Natural Convection – Importance of Buoyancy

- Without buoyancy, heat transfer between a hot (or cold) surface and the fluid surrounding it would be by conduction instead of by natural convection

- **In natural convection heat transfer studies, the primary variable is temperature, and it is desirable to express the net buoyancy force in terms of temperature differences**

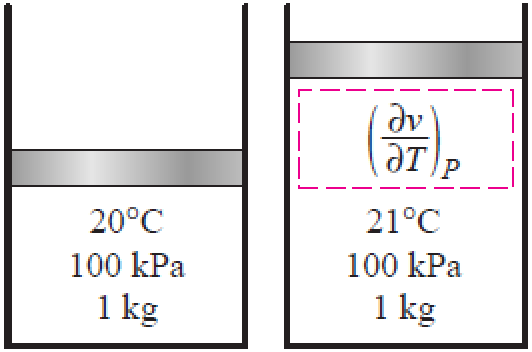
- Volume expansion coefficient β, represents variation of density with temperature at constant pressure

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \qquad (1/\mathrm{K})$$

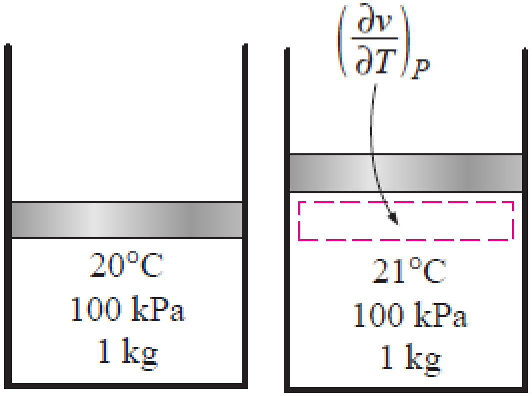
where ρ is the density and T is the temperature of the quiescent fluid away from the surface

$$\beta \approx -\frac{1}{\rho}\frac{\Delta\rho}{\Delta T} = -\frac{1}{\rho}\frac{\rho_\infty - \rho}{T_\infty - T} \qquad (\text{at constant } P)$$

$$\rho_\infty - \rho = \rho\beta(T - T_\infty) \qquad (\text{at constant } P)$$



(*a*) A substance with a large β



(*b*) A substance with a small β

A large β value of for a fluid means a large change in density with temperature, and that the product β*ΔT* represents the fraction of volume change of a fluid that corresponds to a temperature change Δ*T at constant pressure*

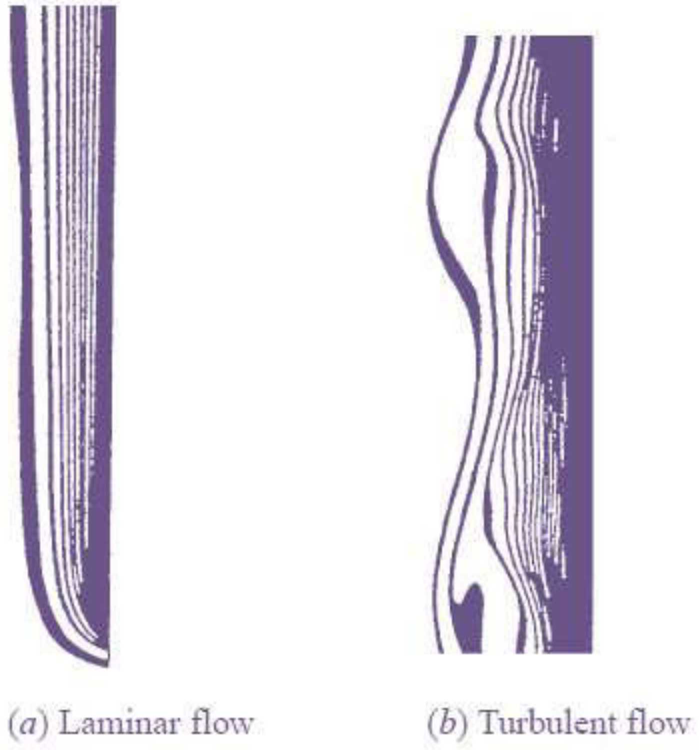
Natural Convection – Importance of Buoyancy

- ***The larger the temperature difference between the fluid adjacent to a hot (or cold) surface and the fluid away from it, the larger the buoyancy force and the stronger the natural convection currents, and thus the higher the heat transfer rate***

- The magnitude of the natural convection heat transfer between a surface and a fluid is directly related to the *flow rate of the fluid.* The flow rate in natural convection is established by the dynamic balance of *buoyancy and friction*

- The friction force increases as more and more solid surfaces are introduced, seriously disrupting the fluid flow and heat transfer

- Most heat transfer correlations in natural convection are based on experimental measurements using instrument *Mach – Zehnder interferometer which gives a plot of isotherms in the vicinity of a surface*

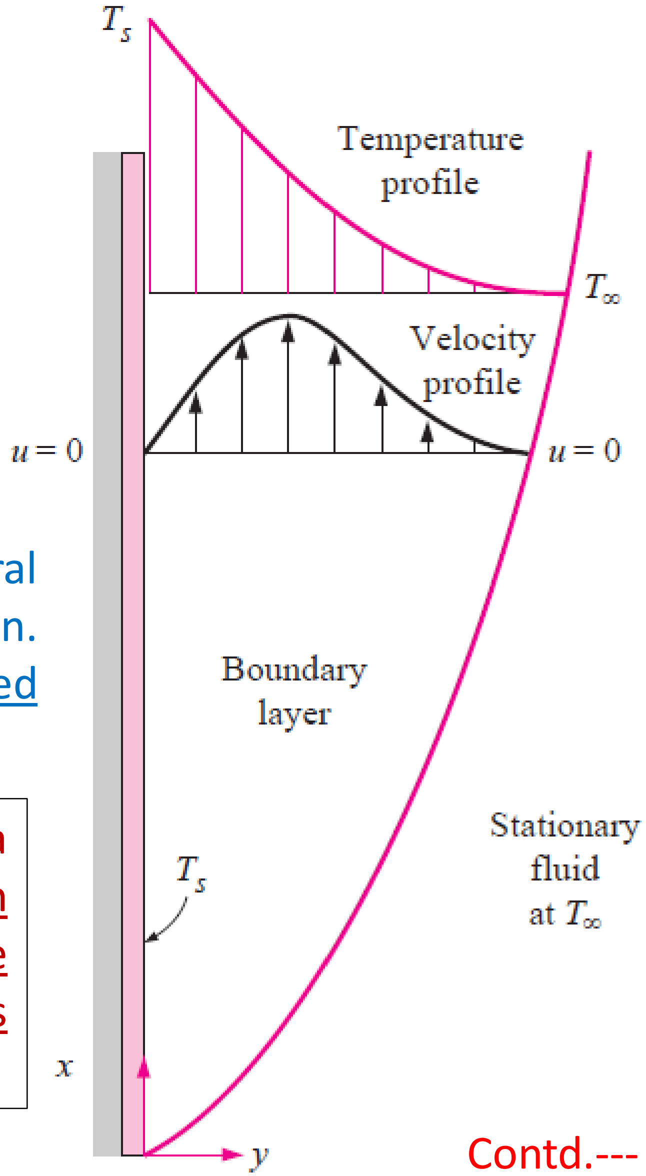


(*a*) Laminar flow (*b*) Turbulent flow

Isotherms in natural convection over a hot plate in air

Natural Convection – Equation of motion

Here we derive the equation of motion that governs the natural convection flow in <u>laminar boundary layer</u>



Continuity equation in natural convection:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Energy equation in natural convection:

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

The conservation of mass and energy equations in natural convection are same as that of forced convection. <u>However the momentum equation needs to be modified to incorporate buoyancy</u>

Consider a vertical hot flat plate immersed in a quiescent fluid body. <u>We assume the natural convection flow to be steady, laminar, two-dimensional, and the fluid to be Newtonian with constant properties including density</u>

Contd.---

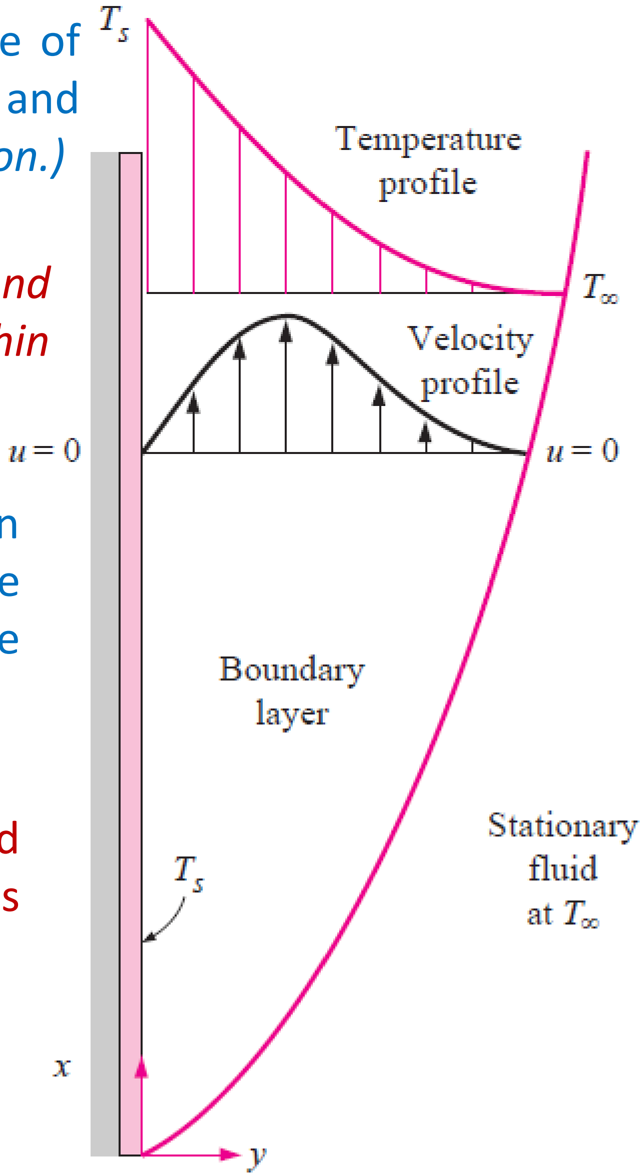
Natural Convection – Equation of motion

- The density difference ρ − ρ_∞ is to be considered since it is this density difference between the inside and the outside of the boundary layer that gives rise to buoyancy force and sustains flow. (This is known as the *Boussinesq approximation.)*

- Gravity acts in the *−x direction. As the flow is steady and* two-dimensional, the *x- and y-components of velocity within boundary layer* are *u = u(x, y) and v = v(x, y), respectively*

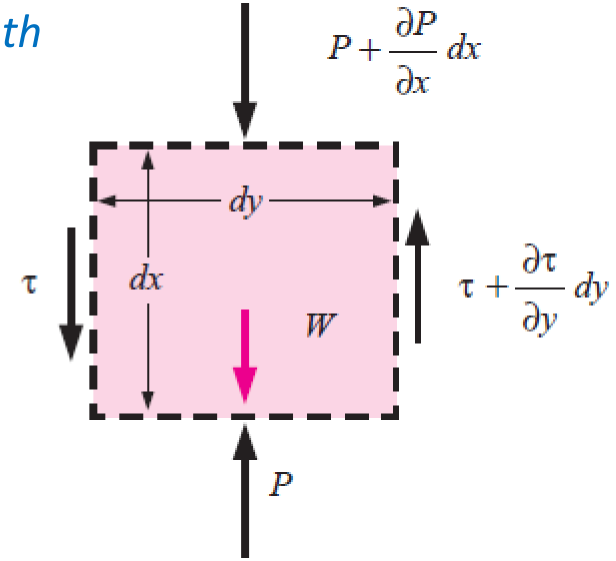
- Velocity and temperature profiles for natural convection over a vertical hot plate shows fluid velocity is zero at the outer edge of velocity boundary layer while temperature gradually decreases to T_∞

- In case of *cold surfaces, the* shape of the velocity and temperature profiles remains the same but their direction is reversed



Natural Convection – Equation of motion derivation

Consider a differential volume element of height dx, length dy, and unit depth in the z-direction (normal to the paper)



Forces acting on a differential control volume in the natural convection boundary layer over a vertical flat plate

Newton's second law of motion for this control volume can be expressed as:

$$\boxed{\delta m \cdot a_x = F_x}$$

Eq.1

The acceleration in the *x-direction is obtained by taking the total differential* of *u(x, y) and* $\delta m = \rho(dx \cdot dy \cdot 1)$

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

Eq.2

The forces acting on the differential volume element in the vertical direction are the pressure forces and shear forces (the normal stresses on top and bottom are small and are ignored

$$\begin{aligned} F_x &= \left(\frac{\partial \tau}{\partial y} dy\right)(dx \cdot 1) - \left(\frac{\partial P}{\partial x} dx\right)(dy \cdot 1) - \rho g(dx \cdot dy \cdot 1) \\ &= \left(\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} - \rho g\right)(dx \cdot dy \cdot 1) \end{aligned}$$

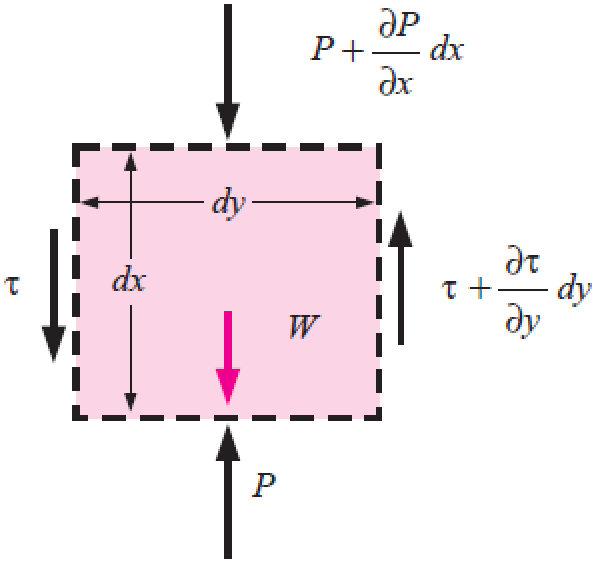
Eq.3

Contd.---

Natural Convection – Equation of motion derivation

$$F_x = \left(\frac{\partial \tau}{\partial y}\, dy\right)(dx \cdot 1) - \left(\frac{\partial P}{\partial x}\, dx\right)(dy \cdot 1) - \rho g(dx \cdot dy \cdot 1)$$
$$= \left(\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} - \rho g\right)(dx \cdot dy \cdot 1)$$

Eq.3



Forces acting on a differential control volume in the natural convection boundary layer over a vertical flat plate

Since $\tau = \mu(\partial u/\partial y)$

*Substituting **Eqs. 2 and 3** into **Eq. 1** and dividing by ρ. dx. dy . 1 gives the conservation of momentum in the x-direction as :*

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} - \rho g$$

Eq.4

The *x-momentum equation in the quiescent fluid outside the boundary layer* can be obtained from the relation above as a special case by setting *u = 0.*

$$\frac{\partial P_\infty}{\partial x} = -\rho_\infty g$$

Eq.5

Velocity component in y direction, the force balance in this direction gives: $\boxed{\partial P/\partial y = 0}$

$v \ll u \qquad \partial v/\partial x \approx \partial v/\partial y \approx 0 \qquad P = P(x) = P_\infty(x)$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} + (\rho_\infty - \rho)g$$

Eq.6

Contd.---

Natural Convection – Equation of motion derivation

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}+(\rho_\infty-\rho)g$$ **Eq.6**

- The last term on R.H.S represents the net upward force per unit volume of the fluid (the difference between the buoyant force and the fluid weight). This is the force that initiates and sustains convection currents.

But we know that $\rho_\infty-\rho=\rho\beta(T-T_\infty)$ (at constant P) **Eq.7**

On substituting Eq.7 in R.H.S of Eq.6 and dividing entire Eq.6 by ρ we get:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)$$ **Eq.8**

The set of three partial differential equations (the continuity, momentum, and the energy equations) that govern natural convection flow over vertical isothermal plates can be reduced to a set of two ordinary nonlinear differential equations by the introduction of a similarity variable

GRASHOF NUMBER

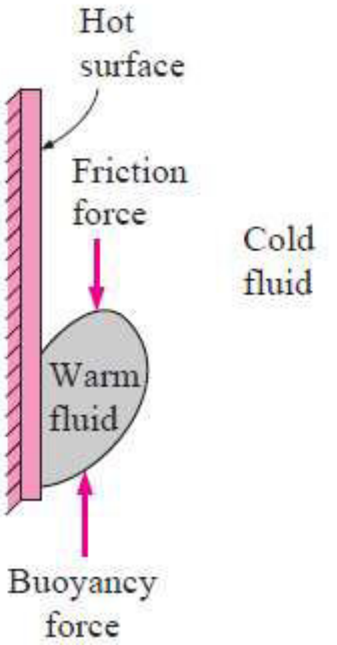
The governing equations of natural convection and the boundary conditions can be nondimensionalized by dividing all dependent and independent variables by suitable constant quantities: all lengths by a characteristic length L_c, all velocities $\mathcal{V}$ at arbitrary velocity

$$x^* = \frac{x}{L_c} \qquad y^* = \frac{y}{L_c} \qquad u^* = \frac{u}{\mathcal{V}} \qquad v^* = \frac{v}{\mathcal{V}} \qquad \text{and} \qquad T^* = \frac{T - T_\infty}{T_s - T_\infty}$$

The non dimensional form of free convection equation is given by:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \left[\frac{g\beta(T_s - T_\infty)L_c^3}{v^2}\right] \frac{T^*}{\mathrm{Re}_L^2} + \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}} \qquad \textbf{Eq.9}$$

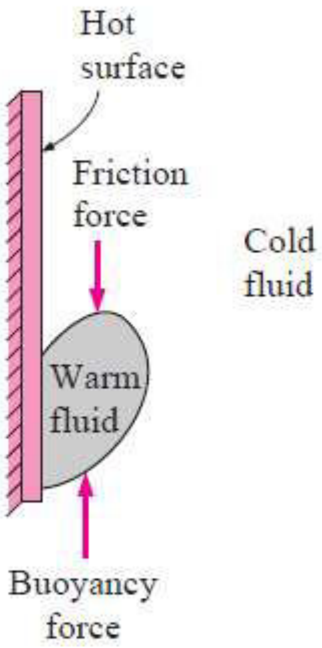
The dimensionless parameter in the brackets represents the natural convection effects, and is called the Grashof number Gr_L. It represents the ratio of buoyancy force to viscous forces acting on the fluid.



GRASHOF NUMBER

$$\mathrm{Gr}_L = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

g = gravitational acceleration, m/s^2
β = coefficient of volume expansion, 1/K ($\beta = 1/T$ for ideal gases)
T_s = temperature of the surface, °C
T_∞ = temperature of the fluid sufficiently far from the surface, °C
L_c = characteristic length of the geometry, m
ν = kinematic viscosity of the fluid, m^2/s



The role played by the Reynolds number in forced convection is played by the Grashof number in natural convection and it provides a criterion to determine whether a flow is laminar or turbulent in natural convection

- For vertical plates, for example, the critical Grashof number is observed to be about 10^9

- When a surface is subjected to external flow, the problem involves both natural and forced convection. The relative importance of each mode of heat transfer is determined by the value of the coefficient Gr_L/Re^2_L

- **Natural convection effects are negligible if $Gr_L/Re^2_L \ll 1$**

- **Forced convection effects are negligible if $Gr_L/Re^2_L \gg 1$**

- **Both free and forced convection effects are dominant if Gr_L/Re^2_L ~~ 1**

Natural Convection over surfaces

- Natural convection heat transfer on a surface depends on the geometry of the surface as well as its orientation, variation of temperature on the surface, thermo physical properties of the fluid

- The complexities of fluid motion make it very difficult to obtain simple analytical relations for heat transfer by solving the governing equations of motion and energy

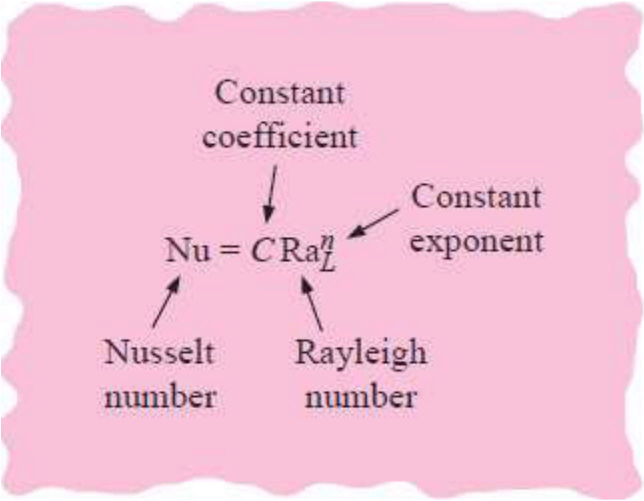
- With the exception of some simple cases, heat transfer relations in natural convection are based on experimental studies

The simple empirical correlations for the average Nusselt number, Nu in natural convection are of the form:

$$\mathrm{Nu} = \frac{hL_c}{k} = C(\mathrm{Gr}_L\,\mathrm{Pr})^n = C\,\mathrm{Ra}_L^n$$

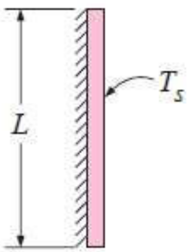
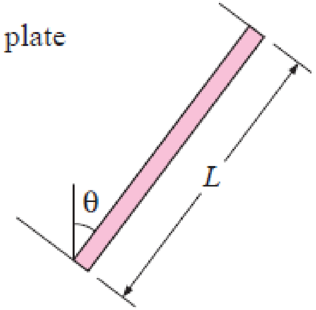
Where Rayleigh number is the product of Grashof and Prandtl numbers

$$\mathrm{Ra}_L = \mathrm{Gr}_L\,\mathrm{Pr} = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}\,\mathrm{Pr}$$



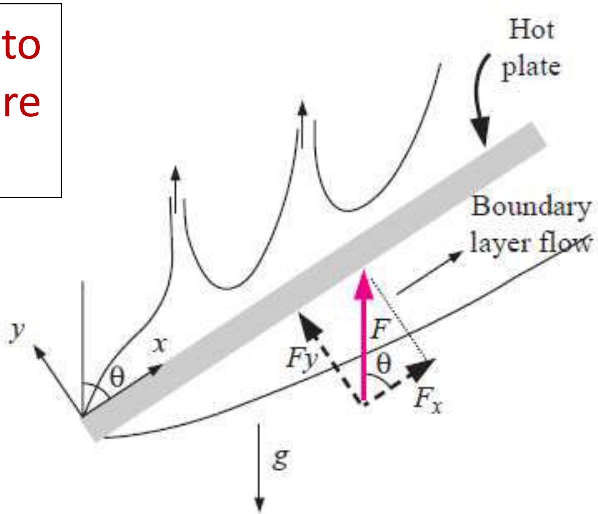
The values of the constants C and n depend on the geometry of the surface and the flow regime, which is characterized by the range of the Rayleigh number

Empirical correlations for the average Nusselt number for natural convection over surfaces

Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical plate 	L	10^4–10^9 10^9–10^{13} Entire range	$\text{Nu} = 0.59\text{Ra}_L^{1/4}$ $\text{Nu} = 0.1\text{Ra}_L^{1/3}$ $\text{Nu} = \left\{0.825 + \dfrac{0.387\text{Ra}_L^{1/6}}{[1 + (0.492/\text{Pr})^{9/16}]^{8/27}}\right\}^2$ (complex but more accurate)
Inclined plate 	L		Use vertical plate equations for the upper surface of a cold plate and the lower surface of a hot plate Replace g by $g \cos\theta$ for $\text{Ra} < 10^9$

Heat transfer on an inclined plate is complex due to resolution of forces on an inclined plane which are bound to alter boundary layer formation

Nusselt number can be determined from the vertical plate relations provided that *g in the Rayleigh number relation is* replaced by *g cosθ for θ < 60*



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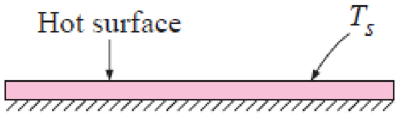
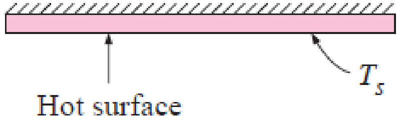
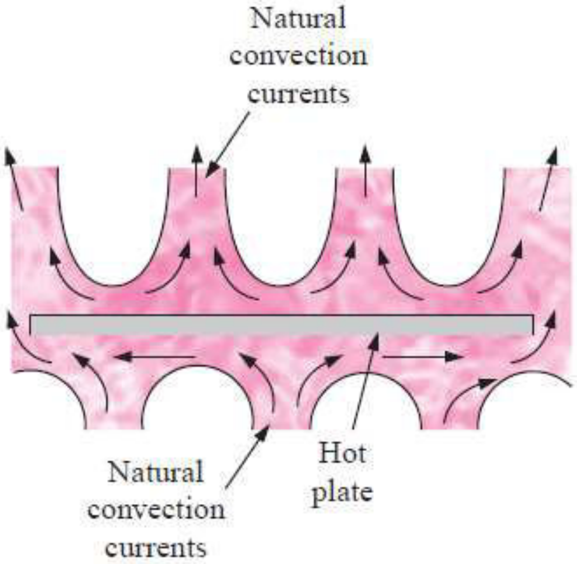
Empirical correlations for the average Nusselt number for natural convection over surfaces

Geometry	Characteristic length L_c	Range of Ra	Nu
Horiontal plate (Surface area A and perimeter p) (*a*) Upper surface of a hot plate (or lower surface of a cold plate) 	A_s/p	10^4–10^7 10^7–10^{11}	$\text{Nu} = 0.54\text{Ra}_L^{1/4}$ $\text{Nu} = 0.15\text{Ra}_L^{1/3}$
(*b*) Lower surface of a hot plate (or upper surface of a cold plate) 		10^5–10^{11}	$\text{Nu} = 0.27\text{Ra}_L^{1/4}$

The rate of heat transfer to or from a horizontal surface depends on whether the surface is facing upward or downward

The characteristic length for horizontal surfaces is calculated from:

$$L_c = \frac{A_s}{p}$$



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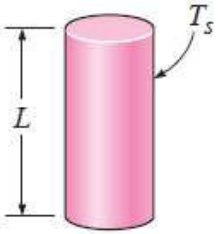
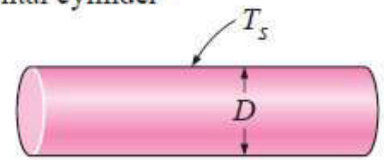
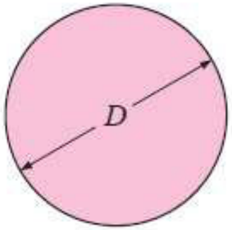
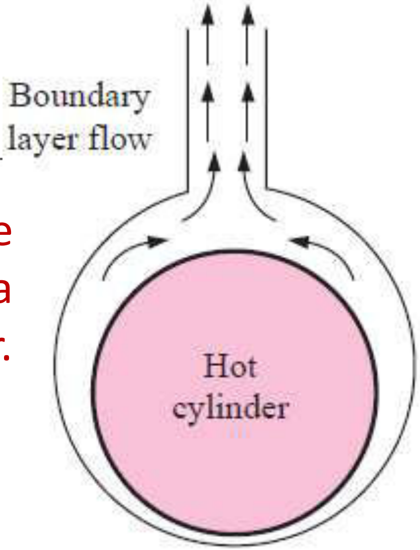
Empirical correlations for the average Nusselt number for natural convection over surfaces

Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical cylinder 	L		A vertical cylinder can be treated as a vertical plate when $D \geq \frac{35L}{\mathrm{Gr}_L^{1/4}}$
Horizontal cylinder 	D	$\mathrm{Ra}_D \leq 10^{12}$	$\mathrm{Nu} = \left\{0.6 + \frac{0.387\mathrm{Ra}_D^{1/6}}{[1 + (0.559/\mathrm{Pr})^{9/16}]^{8/27}}\right\}^2$
Sphere 	D	$\mathrm{Ra}_D \leq 10^{11}$ $(\mathrm{Pr} \geq 0.7)$	$\mathrm{Nu} = 2 + \frac{0.589\mathrm{Ra}_D^{1/4}}{[1 + (0.469/\mathrm{Pr})^{9/16}]^{4/9}}$

The boundary layer over a hot horizontal cylinder start to develop at the bottom, increasing in thickness along the circumference, and forming a rising. Local Nu is highest at the bottom and lowest at the top of cylinder. The opposite is true for cold horizontal cylinder in the warmer medium



HEAT TRANSFER - 80329

MODULE - IV

NATURAL CONVECTION, BOILING AND CONDENSATION

MALLA REDDY ENGINEERING COLLEGE
MREC

Boiling

Part 2

Dr. Satish Idury
Associate Professor

Boiling - Introduction

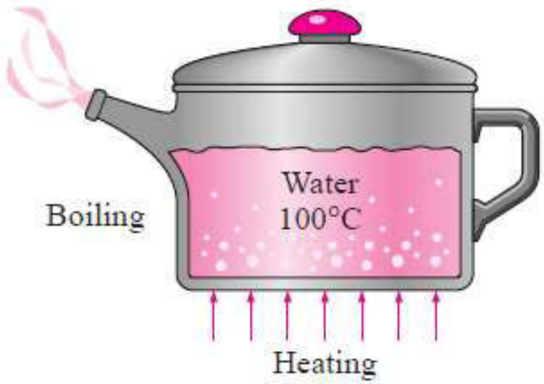
- Boiling is a liquid-to-vapor phase change process just like evaporation, but there are significant differences between the two

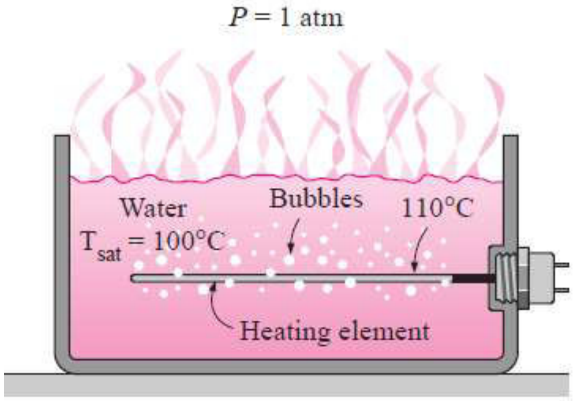
- Evaporation occurs at the liquid–vapor interface when the vapor pressure is less than the saturation pressure of the liquid at a given temperature and does not involve bubble formation or bubble motion

- Boiling, occurs at the solid–liquid interface when a liquid is brought into contact with a surface maintained at a temperature T_s sufficiently above the saturation temperature T_{sat} of the liquid. Boiling is a complicated phenomenon because of the large number of variables involved in the process and the complex fluid motion patterns caused by the bubble formation and growth

Boiling heat flux from a solid surface to the fluid is expressed as:

$$\boxed{\dot{q}_{\text{boiling}} = h(T_s - T_{\text{sat}}) = h\Delta T_{\text{excess}} \qquad (\text{W/m}^2)}$$





Boiling - Introduction

- h_{fg} represents the energy absorbed as a unit mass of liquid vaporizes at a specified temperature or pressure and is the primary quantity of energy during boiling heat transfer

- Bubbles owe their existence to the *surface-tension at the liquid–vapor interface* due to the attraction force on molecules at the interface toward the liquid phase

- The boiling processes in practice do not occur under *equilibrium conditions,* and the bubbles are generally not in thermodynamic equilibrium with the surrounding liquid

- The pressure difference between the liquid and the vapor is balanced by the surface tension at the interface

- The temperature difference between the vapor in a bubble and the surrounding liquid is the driving force for heat transfer between the two phases

Boiling - Classification

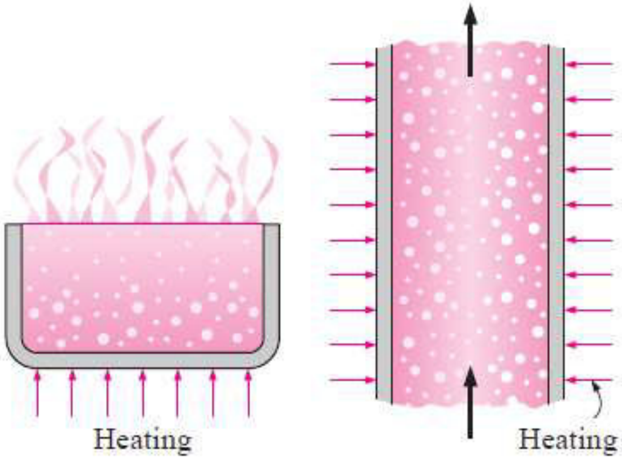
- Boiling is classified as pool boiling or flow boiling, depending on the presence of bulk fluid motion

- Boiling is called pool boiling in the absence of bulk fluid flow and flow boiling (or forced convection boiling) in the presence of it

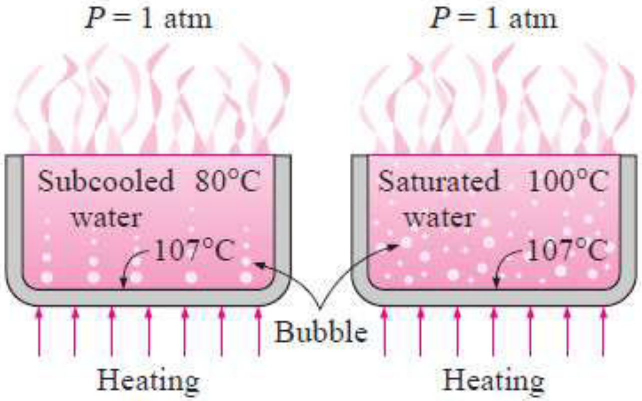
- In pool boiling any motion of fluid is due to natural convection currents while in flow boiling fluid is forced by means of an external device like pump

- Pool and flow boiling are further classified as subcooled boiling or saturated boiling, depending on the bulk liquid temperature.

- Boiling is said to be subcooled (or *local) when the temperature of the main body of* the liquid is below the saturation temperature T_{sat} and saturated (or *bulk) when the temperature of the liquid is* equal to T_{sat}



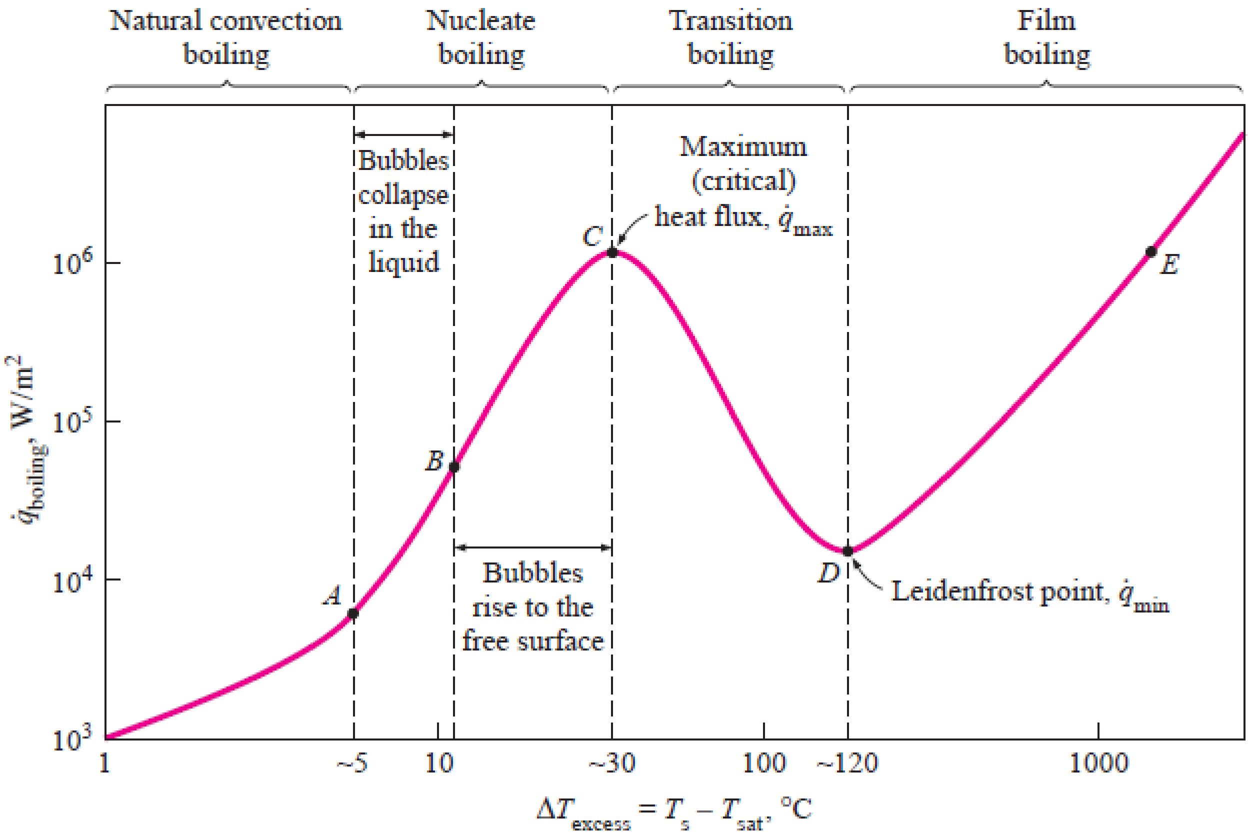
(*a*) Pool boiling (*b*) Flow boiling



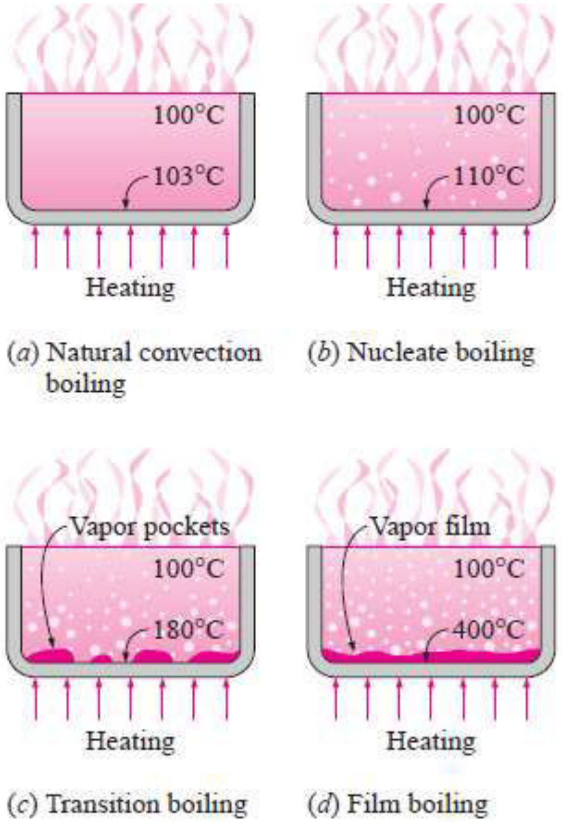
(*a*) Subcooled boiling (*b*) Saturated boiling

Boiling regimes and boiling curve

- ➢Boiling takes different forms, depending on the value of the excess temperature ΔT_{excess}
- ➢The general shape of the boiling curve remains the same for different fluids
- ➢The specific shape of the curve depends on the fluid–heating surface material combination and the fluid pressure, but it is practically independent of the geometry of the heating surface



Typical boiling curve of water at 1 atm pressure



(*a*) Natural convection boiling

(*b*) Nucleate boiling

(*c*) Transition boiling

(*d*) Film boiling

Contd.-----

Boiling regimes and boiling curve

Natural Convection Boiling (to Point A on the Boiling Curve)

- A pure substance at a specified pressure starts boiling when it reaches the saturation temperature at that pressure. But in practice bubble formation will happen only after a certain superheat is provided

- The fluid motion in this mode of boiling is governed by natural convection currents, and heat transfer from the heating surface to the fluid is by natural convection

Nucleate Boiling (between Points A and C)

- The first bubbles start forming at point A of the boiling curve at various preferential sites on the heating surface. The bubbles form at an increasing number of nucleation sites as we move along the boiling curve toward point C

- In region A–B, isolated bubbles are formed at various preferential nucleation sites on the heated surface. But these bubbles are dissipated in the liquid shortly after they separate from the surface

- In region B–C, the heater temperature is further increased, bubbles form at such great rates at such a large number of nucleation sites that they form numerous continuous columns of vapor in the liquid

Contd.-----

Boiling regimes and boiling curve

- At large values of ΔT_{excess}, the rate of evaporation at the heater surface reaches such high values that a large fraction of the heater surface is covered by bubbles, making it difficult for the liquid to reach the heater surface and wet it

- The heat flux increases at a lower rate with increasing ΔT_{excess}, and reaches a maximum at point C. The heat flux at this point is called critical heat flux , $\dot{q}_{max}$

- Nucleate boiling is the most desirable boiling regime in practice because high heat transfer rates can be achieved in this regime with relatively small values of ΔT_{excess}, typically under 30°C for water

Transition Boiling (between Points C and D on the Boiling Curve)

- As the ΔT_{excess} is increased past point C, the heat flux decreases. A large fraction of the heater surface is covered by a vapor film, which acts as an insulation due to the low thermal conductivity of the vapor relative to that of the liquid

- Operation in the transition boiling regime, which is also called the unstable film boiling regime, is avoided in practice

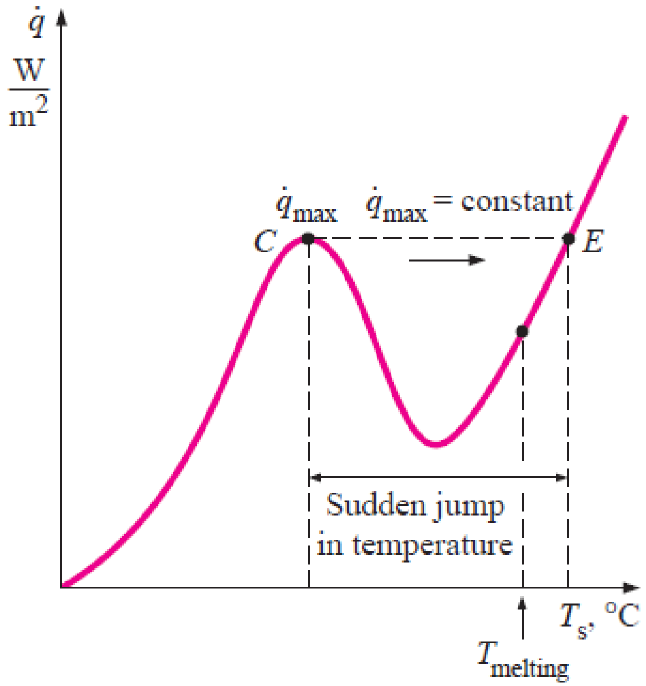
Contd.-----

Boiling regimes and boiling curve

Film Boiling (beyond Point D)

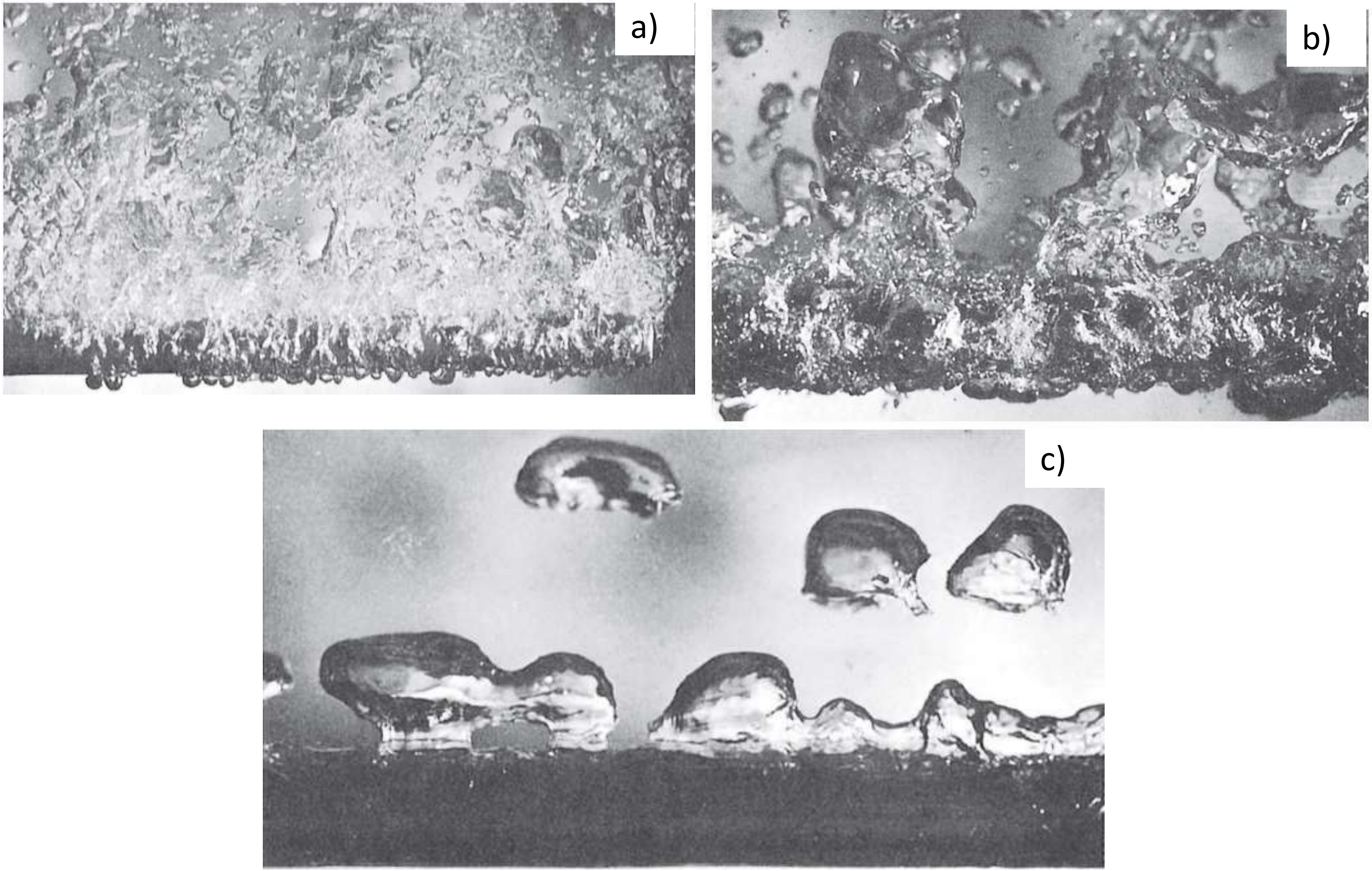
- In this region the heater surface is completely covered by a continuous stable vapor film. The presence of a vapor film between the heater surface and the liquid is responsible for the low heat transfer rates in the film boiling region

- The heat transfer rate increases with increasing excess temperature as a result of heat transfer from the heated surface to the liquid through the vapor film by radiation, which becomes significant at high temperatures



Point C on the boiling curve is also called the burnout point, and the heat flux at this point the burnout heat flux

Various boiling regimes during boiling of methanol on a horizontal 1-cm-diameter steam-heated copper tube



(a) nucleate boiling, (b) transition boiling, and (c) film boiling

Heat transfer correlations in Pool boiling

- Boiling regimes discussed above differ considerably in their character, and thus different heat transfer relations need to be used for different boiling regimes

- In the natural convection boiling regime, boiling is governed by natural convection currents, and heat transfer rates in this case can be determined accurately using natural convection relations

Nucleate Boiling

In the nucleate boiling regime, the rate of heat transfer strongly depends on the nature of nucleation (the number of active nucleation sites on the surface, the rate of bubble formation at each site, etc.), which is difficult to predict. The type and condition of heated surface also affect heat transfer

Empirical correlation proposed by Rohsenow is widely used in nucleation boiling regime:

$$\dot{q}_{\text{nucleate}} = \mu_l h_{fg}\left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2}\left[\frac{C_p(T_s - T_{\text{sat}})}{C_{sf}\, h_{fg}\, \mathrm{Pr}_l^n}\right]^3$$

Contd.-----

Heat transfer correlations in Pool boiling

$$\dot{q}_{\text{nucleate}} = \mu_l\, h_{fg}\left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2}\left[\frac{C_p(T_s - T_{\text{sat}})}{C_{sf}\, h_{fg}\, \text{Pr}_l^n}\right]^3$$

$\dot{q}_{\text{nucleate}}$ = nucleate boiling heat flux, W/m^2
μ_l = viscosity of the liquid, kg/m · s
h_{fg} = enthalpy of vaporization, J/kg
g = gravitational acceleration, m/s^2
ρ_l = density of the liquid, kg/m^3
ρ_v = density of the vapor, kg/m^3
σ = surface tension of liquid–vapor interface, N/m
C_{pl} = specific heat of the liquid, J/kg · °C
T_s = surface temperature of the heater, °C
T_{sat} = saturation temperature of the fluid, °C
C_{sf} = experimental constant that depends on surface–fluid combination
Pr_l = Prandtl number of the liquid
n = experimental constant that depends on the fluid

C_{sf} *values for various fluid–surface combinations* can be used for *any geometry. It is found that the* rate of heat transfer during nucleate boiling is essentially independent of the geometry and orientation of the heated surface

Contd.-----

Heat transfer correlations in Pool boiling

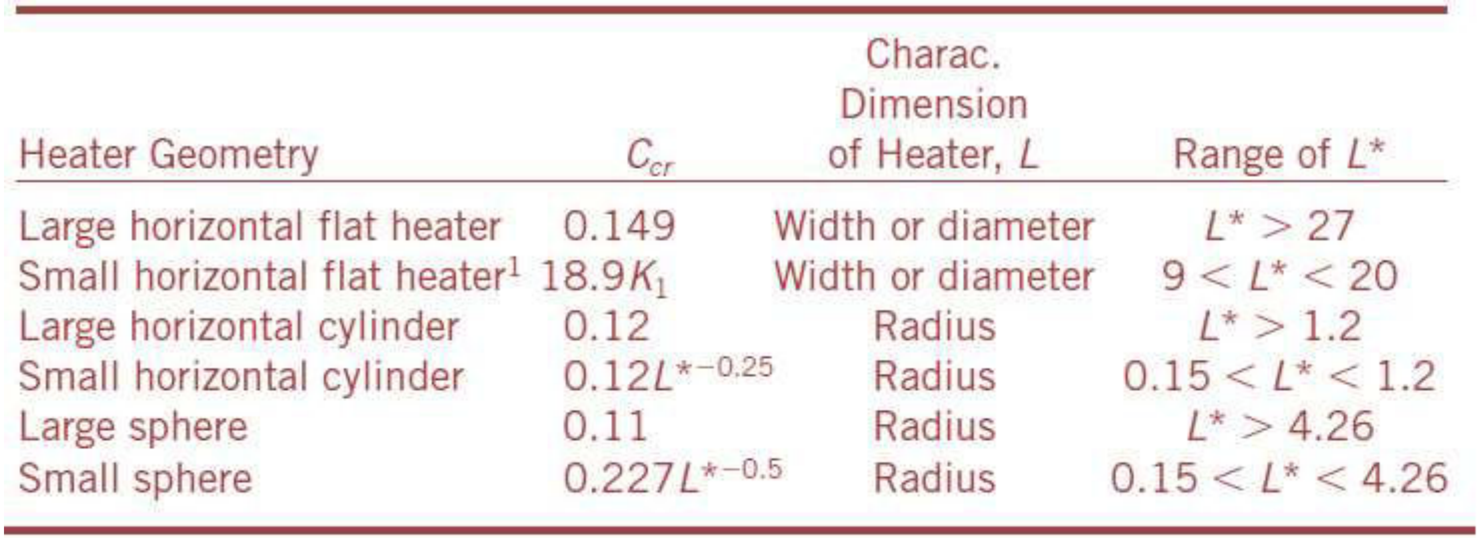
Peak heat flux:

In the design of boiling heat transfer equipment, it is extremely important for the designer to have a knowledge of the maximum heat flux in order to avoid the danger of burnout. The maximum (or critical) heat flux in nucleate pool boiling was determined theoretically

$$\dot{q}_{\max} = C_{cr}\, h_{fg}[\sigma g \rho_v^2 (\rho_l - \rho_v)]^{1/4}$$

ρ_v increases but and h_{fg} decrease with increasing pressure, and thus the change in q max with pressure depends on which effect dominates

(dimensionless parameter $L^* = L[g(\rho_l - \rho_v)/\sigma]^{1/2}$) is to be calculated for estimating C_{cr}



Heater Geometry	C_{cr}	Charac. Dimension of Heater, L	Range of L^*
Large horizontal flat heater	0.149	Width or diameter	$L^* > 27$
Small horizontal flat heater[1]	$18.9K_1$	Width or diameter	$9 < L^* < 20$
Large horizontal cylinder	0.12	Radius	$L^* > 1.2$
Small horizontal cylinder	$0.12L^{*-0.25}$	Radius	$0.15 < L^* < 1.2$
Large sphere	0.11	Radius	$L^* > 4.26$
Small sphere	$0.227L^{*-0.5}$	Radius	$0.15 < L^* < 4.26$

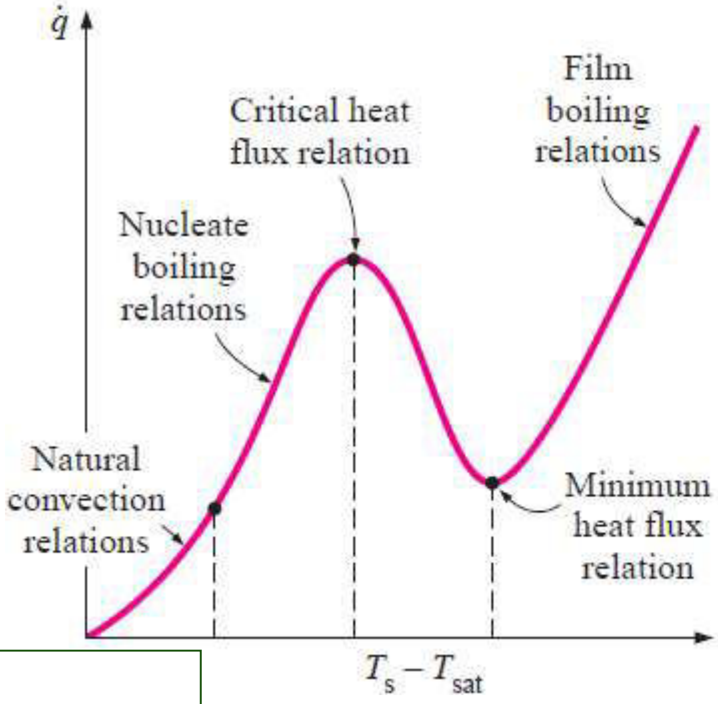
Contd.-----

Heat transfer correlations in Pool boiling

Minimum Heat Flux

- Minimum heat flux, which occurs at the Leidenfrost point, is of practical interest since it represents the lower limit for the heat flux in the film boiling regime

$$\dot{q}_{\text{min}} = 0.09\rho_v h_{fg} \left[\frac{\sigma g(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4}$$



Film Boiling

The heat flux for film boiling on a horizontal cylinder or sphere of diameter D is given by:

k_v is the thermal conductivity of the vapor in W/m · °C

$$C_{\text{film}} = \begin{cases} 0.62 \text{ for horizontal cylinders} \\ 0.67 \text{ for spheres} \end{cases}$$

$$\dot{q}_{\text{film}} = C_{\text{film}} \left[\frac{g k_v^3 \rho_v (\rho_l - \rho_v)[h_{fg} + 0.4 C_{pv} (T_s - T_{\text{sat}})]}{\mu_v D (T_s - T_{\text{sat}})}\right]^{1/4} (T_s - T_{\text{sat}})$$

Vapor properties are evaluated at the film temperature, given as $T_f = (T_s + T_{sat})/2$, the average temperature of the vapor film. The liquid properties and *h_{fg} are to be evaluated at the saturation temperature at the* specified pressure

Contd.-----

Heat transfer correlations in Pool boiling

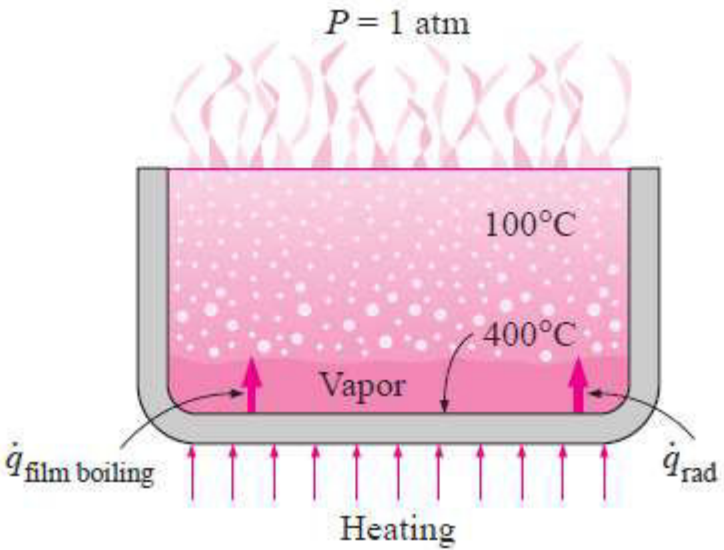
- At high surface temperatures (typically above 300°C), heat transfer across the vapor film by radiation becomes significant and needs to be considered
- Treating the vapor film as a transparent medium sandwiched between two large parallel plates and approximating the liquid as a blackbody, radiation heat transfer can be determined from:

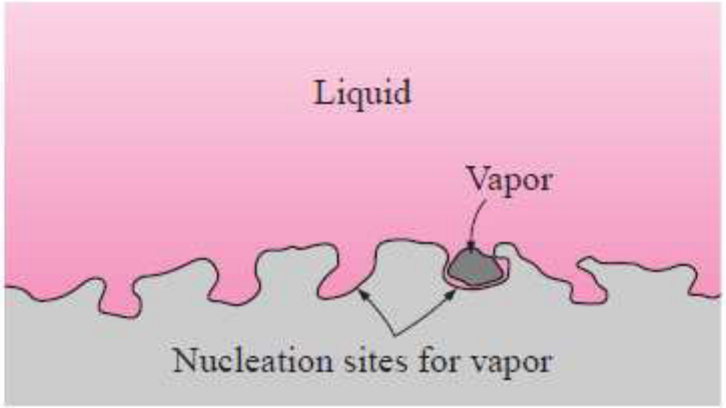
$$\dot{q}_{\text{rad}} = \varepsilon\sigma\left(T_s^4 - T_{\text{sat}}^4\right)$$

- In case of film boiling, radiation and convection mechanisms adversely affect each other:

If $\dot{q}_{\text{rad}} < \dot{q}_{\text{film}}$, $\dot{q}_{\text{total}} = \dot{q}_{\text{film}} + \frac{3}{4}\dot{q}_{\text{rad}}$

Rough surfaces enhances nucleation sites and increases boiling heat transfer





HEAT TRANSFER - 80329

MODULE - IV

NATURAL CONVECTION, BOILING AND CONDENSATION

Condensation

Part 3

Dr. Satish Idury
Associate Professor

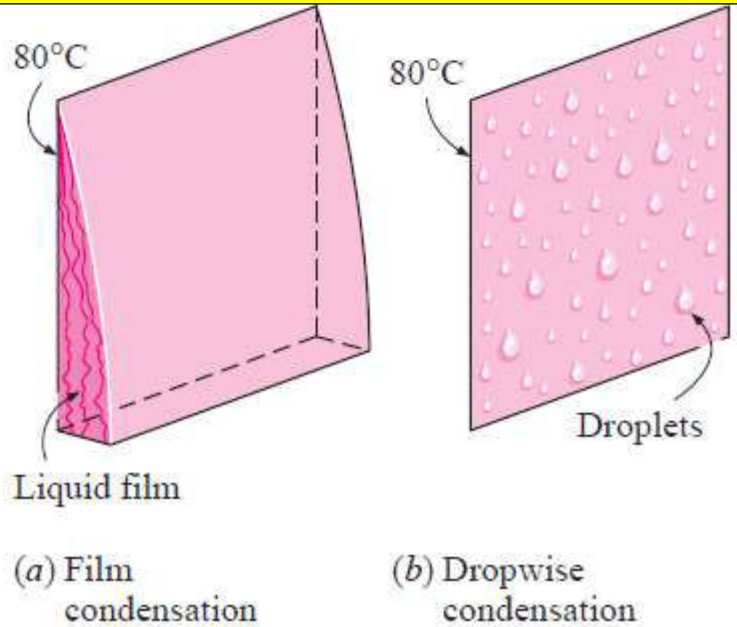
Condensation - Introduction

- Condensation occurs when the temperature of a vapor is reduced below its saturation temperature $T_{sat.}$ It can also occur on the free surface of a liquid or a gas when the liquid or the gas to which the vapor is exposed is below T_{sat}

- In **film condensation, the condensate wets the surface** and forms a liquid film on the surface that slides down under the influence of gravity

- In **dropwise condensation, the condensed vapor forms** droplets on the surface instead of a continuous film, and the surface is covered by countless droplets of varying diameters

- Heat transfer rates are more than 10 times larger than those associated with film condensation can be achieved with dropwise condensation

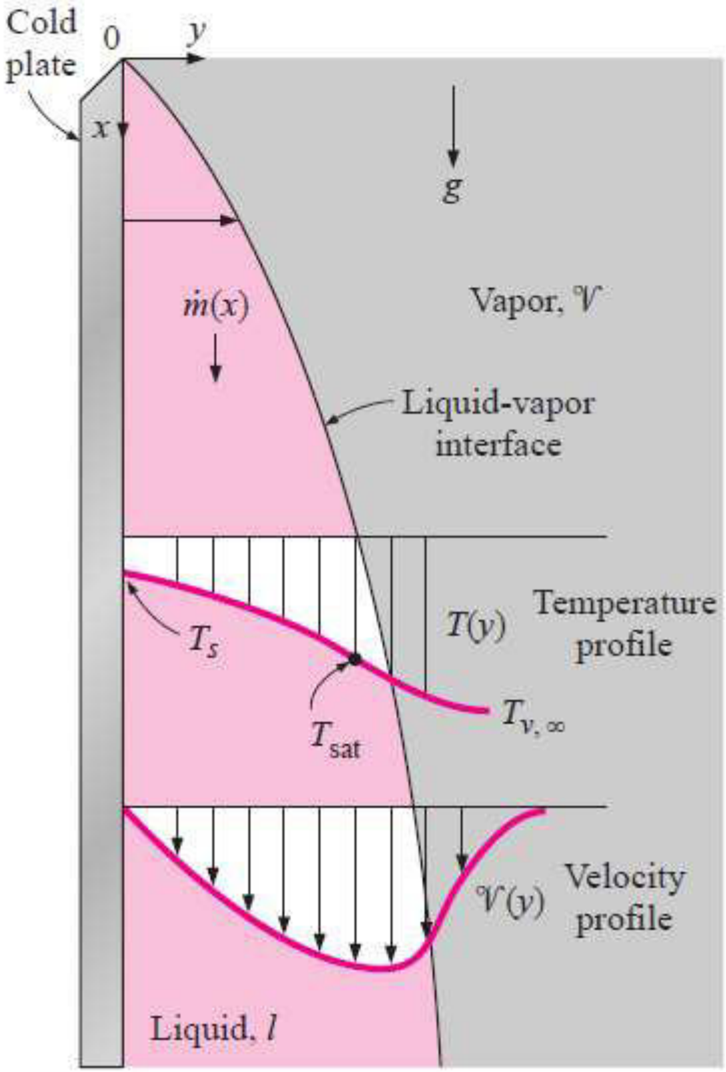


(*a*) Film condensation

(*b*) Dropwise condensation

Film condensation on vertical plate

- The thickness of the film increases in the flow direction x because of continued condensation at the liquid–vapor interface
- h_{fg} (the latent heat of vaporization) is released during condensation and is transferred through the film to the plate surface at temperature T_s
- Velocity of the condensate at the wall is zero because of the "no-slip" condition and reaches a maximum at the liquid–vapor interface
- The temperature of the condensate is T_{sat} at the interface and decreases gradually to T_s at the wall
- Heat transfer in condensation depends on whether the condensate flow is laminar or turbulent



Reynolds number – flow regime for condensation

$$\mathrm{Re} = \frac{D_h \rho_l \mathcal{V}_l}{\mu_l} = \frac{4 A_c \rho_l \mathcal{V}_l}{p \mu_l} = \frac{4 \rho_l \mathcal{V}_l \delta}{\mu_l} = \frac{4 \dot{m}}{p \mu_l} \qquad \textbf{Eq.1}$$

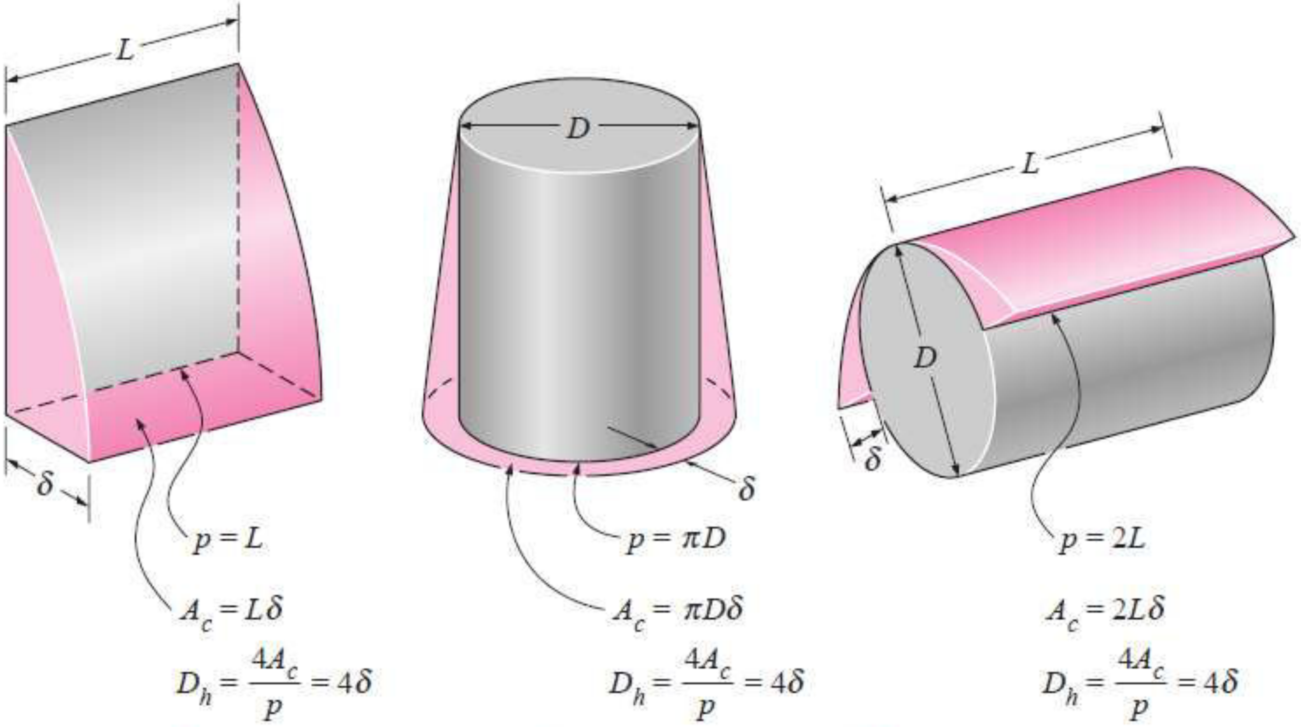
$D_h = 4A_c / p = 4\delta$ = hydraulic diameter of the condensate flow

p = wetted perimeter of the condensate

A_c = p= wetted perimeter X film thickness, m^2

ρ_l = density of the liquid, kg/m^3

μ_l = *viscosity of the liquid, kg/m · s*



(*a*) Vertical plate (*b*) Vertical cylinder (*c*) Horizontal cylinder

Flow regime in condensation

If the condensate is cooled from superheated vapor state at T_v to T_s then obviously heat transfer will be more than h_{fg} and is given by:

$$h^*_{fg} = h_{fg} + 0.68C_{pl}\,(T_{\text{sat}} - T_s) + C_{pv}\,(T_v - T_{\text{sat}})$$

Rate of heat transfer can be expressed as:

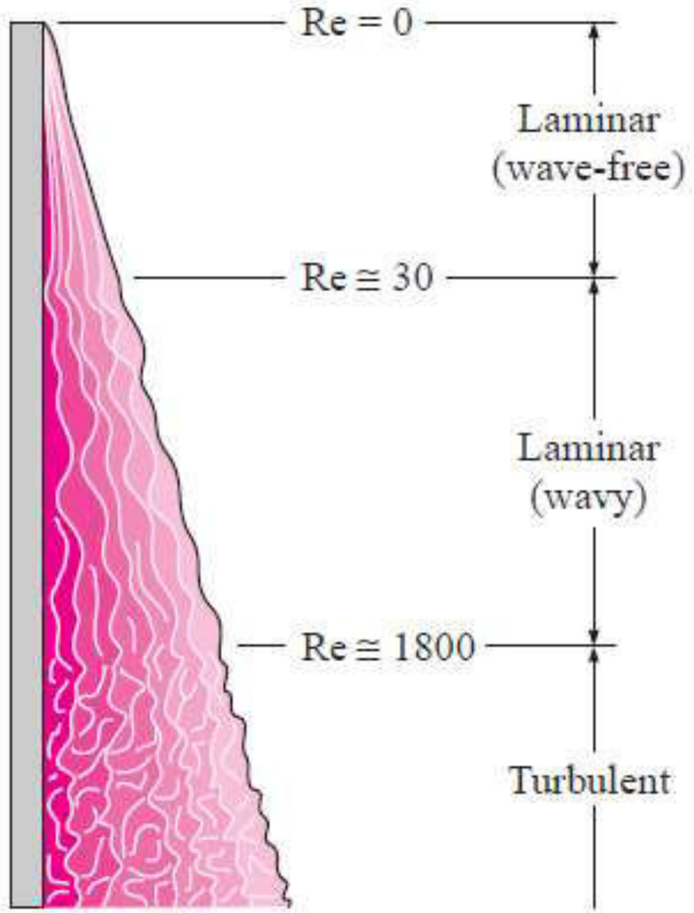
$$\dot{Q}_{\text{conden}} = hA_s(T_{\text{sat}} - T_s) = \dot{m}h^*_{fg} \qquad \textbf{Eq.2}$$

By substituting Eq.2 in Eq.1

$$\text{Re} = \frac{4\dot{Q}_{\text{conden}}}{p\mu_l\, h^*_{fg}} = \frac{4A_s\, h(T_{\text{sat}} - T_s)}{p\mu_l\, h^*_{fg}} \qquad \textbf{Eq.3}$$

➢The temperature of the liquid film varies from T_{sat} on the liquid–vapor interface to T_s at the wall surface. Properties of the liquid should be evaluated at the film temperature $T_f = (T_{sat} + T_s)/2$,

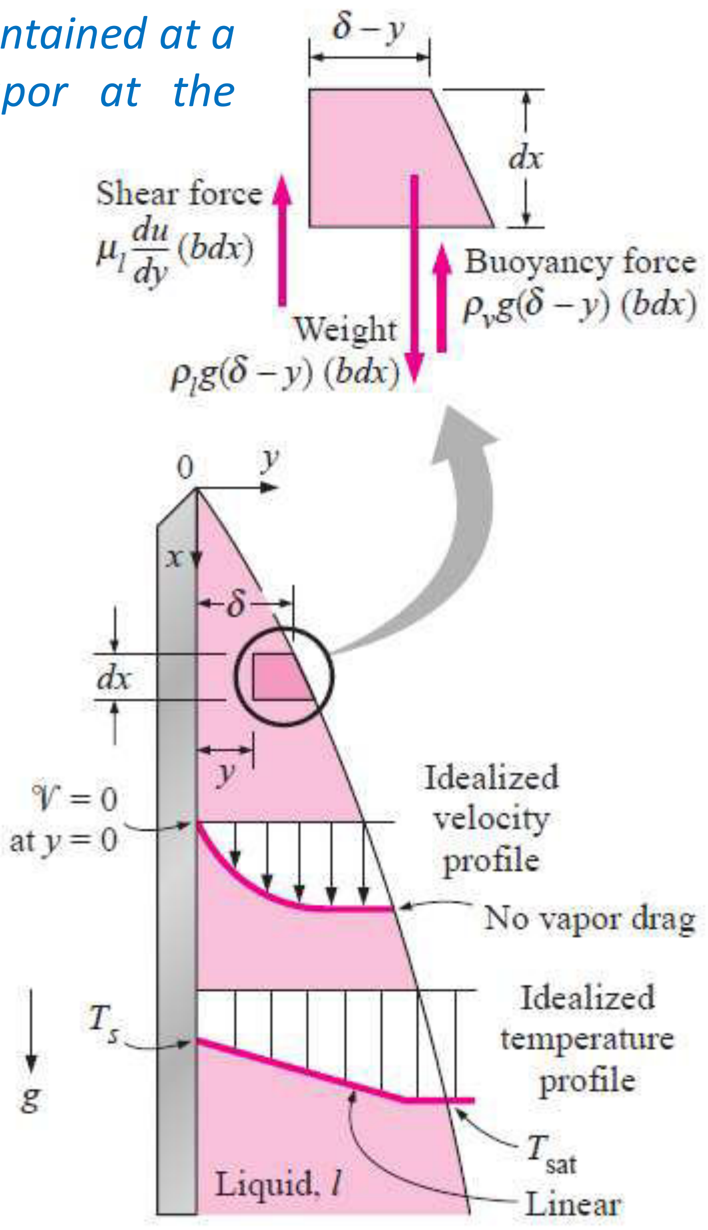
➢h_{fg}, should be evaluated at T_{sat} since it is not affected by the subcooling of the liquid



➢The flow of liquid film exhibits different regimes, depending on the value of the Reynolds number

Nusselt theory on film condensation

Consider a vertical plate of height L and width b maintained at a constant temperature T_s *that is exposed to vapor at the saturation temperature* T_{sat}*,* $T_s < T_{sat}$



Assumptions in the derivation

- Plate and the vapor are maintained at constant temperatures of T_s and T_{sat}, respectively, and the temperature across the liquid film varies linearly
- Heat transfer across the liquid film is by pure conduction
- The velocity of the vapor is low (or zero) so that it exerts no drag on the condensate (no viscous shear on the liquid–vapor interface)
- The flow of the condensate is laminar and the properties of the liquid are constant
- The acceleration of the condensate layer is negligible

Contd.---

Nusselt theory on film condensation

since the acceleration of the fluid is zero

$$\sum F_x = ma_x = 0$$

$$F_{\text{downward}\downarrow} = F_{\text{upward}\uparrow}$$

$$\text{Weight} = \text{Viscous shear force} + \text{Buoyancy force}$$

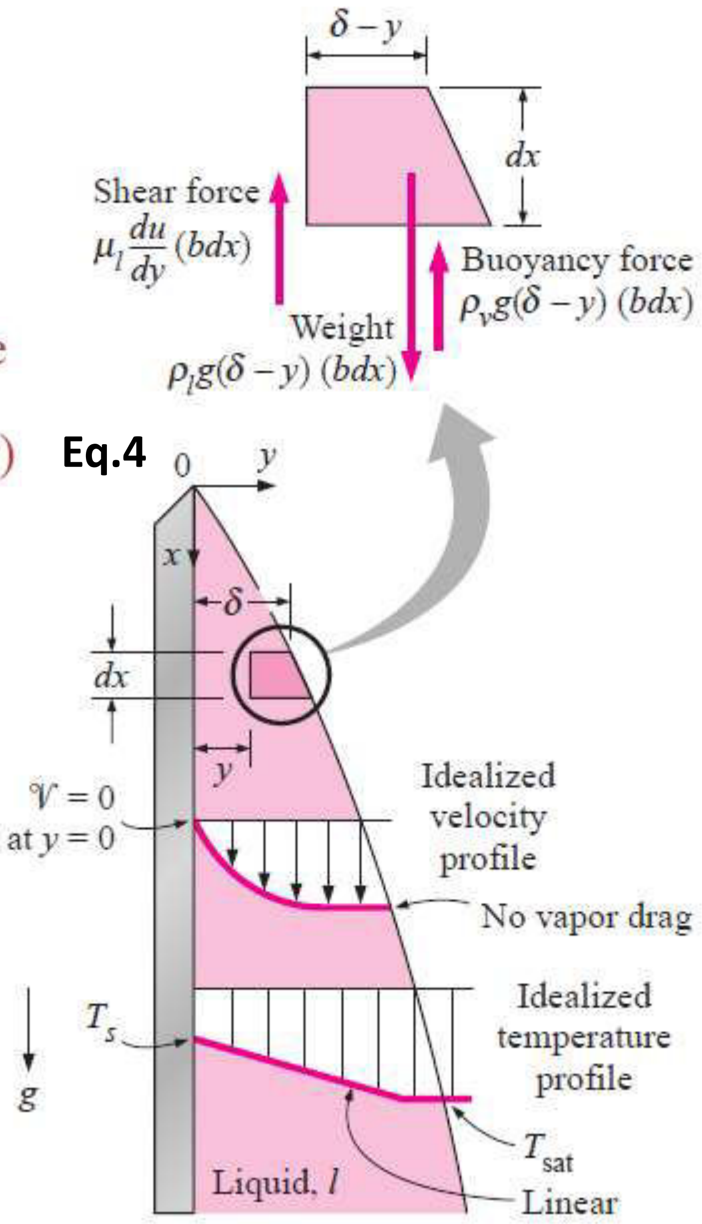
$$\rho_l g(\delta - y)(bdx) = \mu_l \frac{du}{dy}(bdx) + \rho_v g(\delta - y)(bdx)$$ **Eq.4**

Cancelling (bdx) throughout Eq.4 gives

$$\frac{du}{dy} = \frac{(\rho_l - \rho_v)g(\delta - y)}{\mu_l}$$

Integrating from *y = 0 where u = 0 (because of the no-slip boundary condition)* to *y = y where u = u(y) gives:*

$$u(y) = \frac{(\rho_l - \rho_v)g}{\mu_l}\left(y\delta - \frac{y^2}{2}\right)$$ **Eq.5**



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Nusselt theory on film condensation

The mass flow rate of the condensate at a location x, where the boundary layer thickness δ is:

$$\dot{m}(x) = \int_A \rho_l u(y)\,dA = \int_{y=0}^{\delta} \rho_l u(y)\,b\,dy$$ **Eq.6**

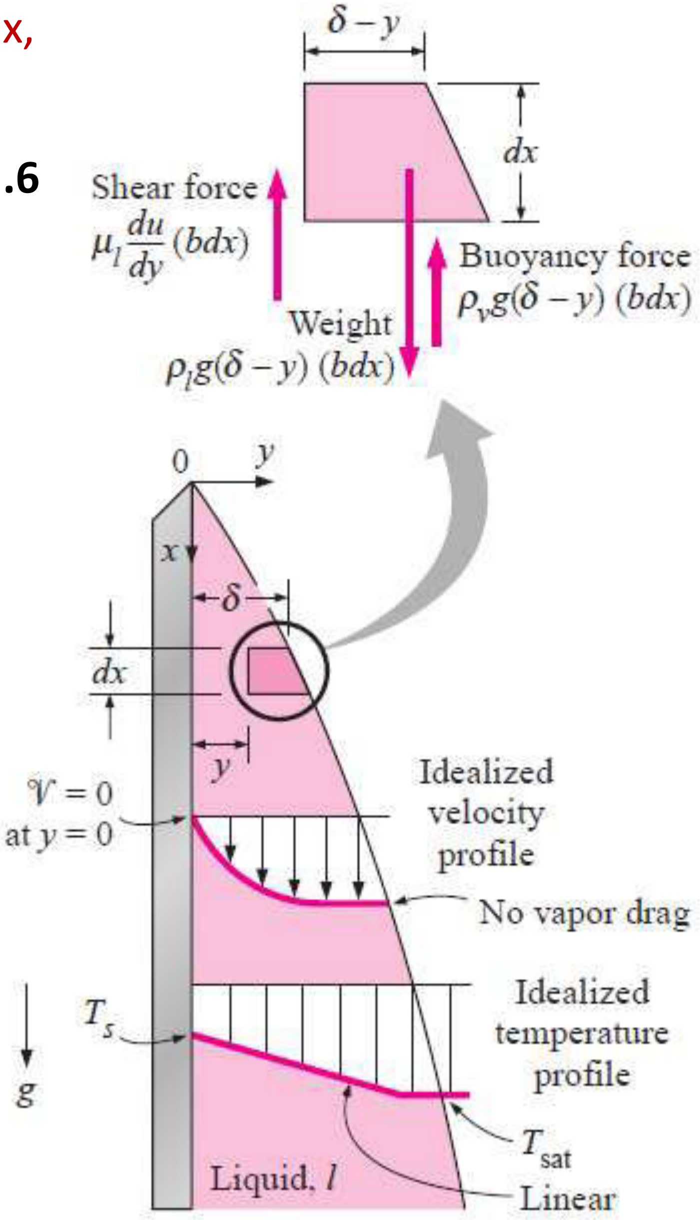
Substituting the u(y) relation from Eq.5 in Eq.6

$$\dot{m}(x) = \frac{gb\rho_l(\rho_l - \rho_v)\delta^3}{3\mu_l}$$ **Eq.7**

whose derivative with respect to x is:

$$\frac{d\dot{m}}{dx} = \frac{gb\rho_l(\rho_l - \rho_v)\delta^2}{\mu_l}\frac{d\delta}{dx}$$ **Eq.8**

which represents the rate of condensation of vapor over a vertical distance dx. The rate of heat transfer from the vapor to the plate through the liquid film is simply equal to the heat released as the vapor is condensed



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Nusselt theory on film condensation

The rate of heat transfer from the vapor to the plate through the liquid film is simply equal to the heat released as the vapor is condensed is:

$$d\dot{Q} = h_{fg}\,d\dot{m} = k_l(bdx)\frac{T_{sat}-T_s}{\delta}$$

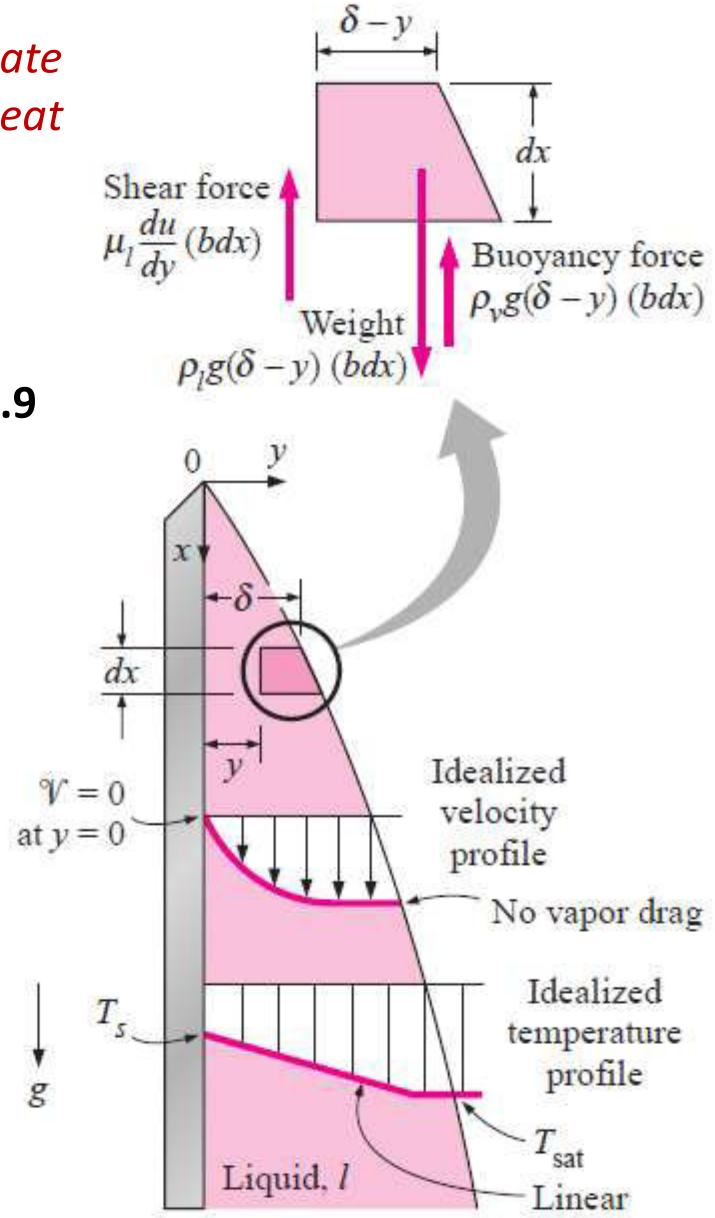
$$\frac{d\dot{m}}{dx} = \frac{k_l\,b}{h_{fg}}\,\frac{T_{sat}-T_s}{\delta}$$

Eq.9

Equating Eq.8 and Eq.9

$$\delta^3\,d\delta = \frac{\mu_l\,k_l(T_{sat}-T_s)}{g\rho_l(\rho_l-\rho_v)h_{fg}}\,dx$$

Eq.10



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Nusselt theory on film condensation

Integrating from x = 0 where δ = 0 (the top of the plate) to x = x where δ= δ(x), the liquid film thickness at any location x is:

$$\delta(x) = \left[\frac{4\mu_l k_l (T_{\text{sat}} - T_s)x}{g\rho_l(\rho_l - \rho_v)h_{fg}}\right]^{1/4}$$

Eq.11

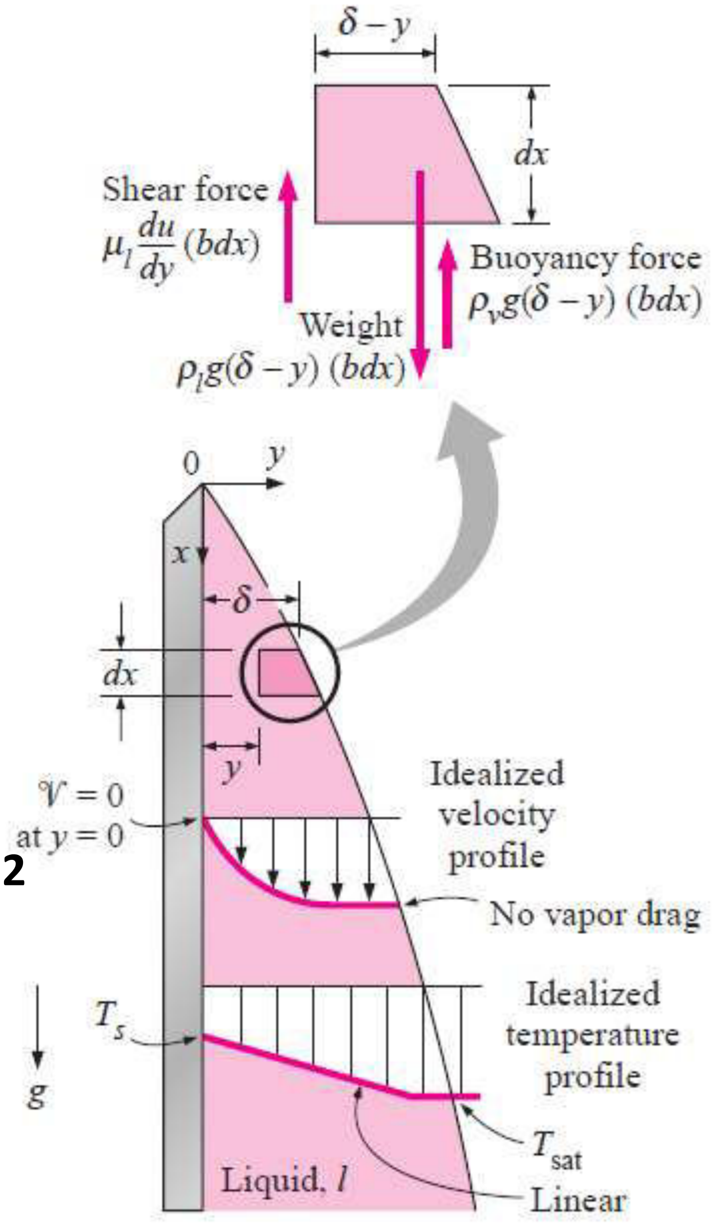
The heat transfer rate from the vapor to the plate at a location x is:

$$\boxed{\dot{q}_x = h_x(T_{\text{sat}} - T_s) = k_l\frac{T_{\text{sat}} - T_s}{\delta}}$$

$$\Downarrow$$

$$\boxed{h_x = \frac{k_l}{\delta(x)}}$$

Eq.12



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Nusselt theory on film condensation

Inserting Eq.12 in Eq.11 the local heat transfer coefficient h_x is

$$h_x = \left[\frac{g\rho_l(\rho_l - \rho_v)h_{fg}\,k_l^3}{4\mu_l\,(T_{\text{sat}} - T_s)x}\right]^{1/4}$$

The average heat transfer coefficient over the entire plate is determined to be:

$$h = h_{\text{ave}} = \frac{1}{L}\int_0^L h_x\,dx = \frac{4}{3}h_{x=L} = 0.943\left[\frac{g\rho_l(\rho_l - \rho_v)h_{fg}\,k_l^3}{\mu_l\,(T_{\text{sat}} - T_s)L}\right]^{1/4}$$

Eq.13

The above expression does not take into account the effects of the nonlinear temperature profile in the liquid film and the cooling of the liquid below the saturation temperature. Both these effects can be accounted by replacing h_{fg} with h^*_{fg}

$$h_{\text{vert}} = 0.943\left[\frac{g\rho_l(\rho_l - \rho_v)h^*_{fg}\,k_l^3}{\mu_l\,(T_{\text{sat}} - T_s)L}\right]^{1/4} \qquad (\text{W/m}^2\cdot{}^\circ\text{C}),\ 0 < \text{Re} < 30$$

Eq.14

Other correlations for Condensation heat transfer

Heat transfer coefficient h_{vert} in terms of Re by substituting Eq.7 in Eq.1:

$$h_{\mathrm{vert}} \cong 1.47 k_l \,\mathrm{Re}^{-1/3} \left(\frac{g}{\nu_l^2}\right)^{1/3}, \qquad \begin{matrix} 0 < \mathrm{Re} < 30 \\ \rho_v \ll \rho_l \end{matrix}$$

Wavy laminar flow on vertical plates:

$$h_{\mathrm{vert,\,wavy}} = \frac{\mathrm{Re}\, k_l}{1.08\,\mathrm{Re}^{1.22} - 5.2} \left(\frac{g}{\nu_l^2}\right)^{1/3}, \qquad \begin{matrix} 30 < \mathrm{Re} < 1800 \\ \rho_v \ll \rho_l \end{matrix}$$

$$\mathrm{Re}_{\mathrm{vert,\,wavy}} = \left[4.81 + \frac{3.70\, L k_l (T_{\mathrm{sat}} - T_s)}{\mu_l\, h_{fg}^*} \left(\frac{g}{\nu_l^2}\right)^{1/3}\right]^{0.820}, \quad \rho_v \ll \rho_l$$

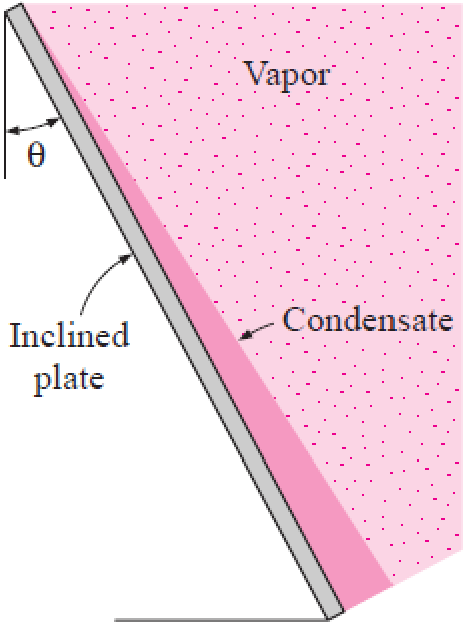
Turbulent flow on vertical plates:

$$h_{\mathrm{vert,\,turbulent}} = \frac{\mathrm{Re}\, k_l}{8750 + 58\,\mathrm{Pr}^{-0.5}\,(\mathrm{Re}^{0.75} - 253)} \left(\frac{g}{\nu_l^2}\right)^{1/3}, \qquad \begin{matrix} \mathrm{Re} > 1800 \\ \rho_v \ll \rho_l \end{matrix}$$

$$\mathrm{Re}_{\mathrm{vert,\,turbulent}} = \left[\frac{0.0690\, L k_l\, \mathrm{Pr}^{0.5}\, (T_{\mathrm{sat}} - T_s)}{\mu_l\, h_{fg}^*} \left(\frac{g}{\nu_l^2}\right)^{1/3} - 151\,\mathrm{Pr}^{0.5} + 253\right]^{4/3}$$

Correlations for Condensation heat transfer for other geometries

Inclined plates:



$$u(y) = \frac{(\rho_l - \rho_v)g}{\mu_l}\left(y\delta - \frac{y^2}{2}\right)$$ **Eq.5**

The above equation developed for vertical plates can also be used for laminar film condensation on the upper surfaces of plates that are inclined by an angle from the vertical, by replacing g in that equation by g cosϑ

$$h_{\text{inclined}} = h_{\text{vert}}\,(\cos\theta)^{1/4} \qquad \text{(laminar)}$$

Vertical tubes:

$$h_{\text{vert}} = 0.943\left[\frac{g\rho_l(\rho_l - \rho_v)h^*_{fg}\,k_l^3}{\mu_l\,(T_{\text{sat}} - T_s)L}\right]^{1/4} \qquad (\text{W/m}^2\cdot{}^\circ\text{C}),\ 0 < \text{Re} < 30$$ **Eq.14**

The above equation for vertical plates can also be used to calculate the average heat transfer coefficient for laminar film condensation on the outer surfaces of vertical tubes provided that the tube diameter is large relative to the thickness of the liquid film

Contd.---

Correlations for Condensation heat transfer for other geometries

Horizontal tubes and spheres:

- Nusselt's analysis of film condensation on vertical plates can also be extended to horizontal tubes and spheres
- The average heat transfer coefficient for film condensation on the outer surfaces of a horizontal tube is determined to be:

$$h_{\text{horiz}} = 0.729\left[\frac{g\rho_l(\rho_l - \rho_v)\,h^*_{fg}\,k_l^3}{\mu_l(T_{\text{sat}} - T_s)D}\right]^{1/4} \qquad (\text{W/m}^2\cdot{}^\circ\text{C})$$

Eq.15

D is the diameter of the horizontal tube. Eq.15 can easily be modified for a sphere by replacing the constant 0.729 by 0.815

A comparison of the heat transfer coefficient relations for a vertical tube of height L and a horizontal tube of diameter D:

$$\frac{h_{\text{vert}}}{h_{\text{horiz}}} = 1.29\left(\frac{D}{L}\right)^{1/4}$$

$$h_{\text{vertical}} = h_{\text{horizontal}} \text{ gives } L = 1.29^4\,D = 2.77D$$

it is common practice to place the tubes in a condenser horizontally to maximize the condensation heat transfer coefficient on the outer surfaces of the tubes

Contd.---

Other factors affecting heat transfer in film condensation

Effect of Vapor Velocity

- When the vapor velocity is high, the vapor will “pull” the liquid at the interface along since the vapor velocity at the interface must drop to the value of the liquid velocity

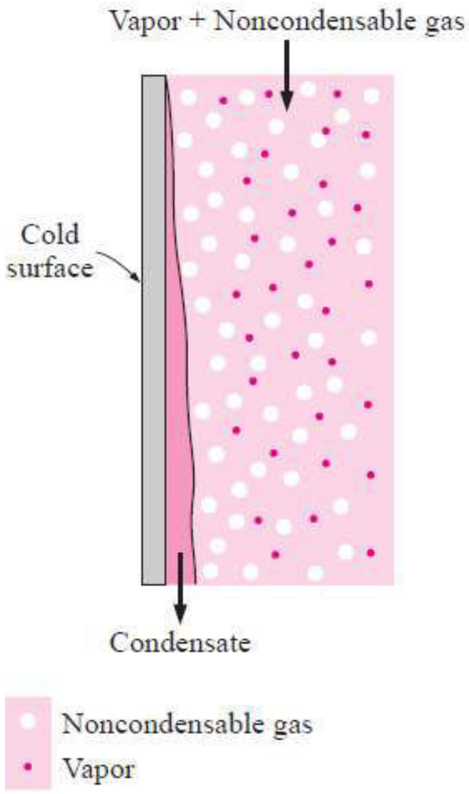
- If the vapor flows downward this additional force will increase the average velocity of the liquid and thus decrease the film thickness. This, in turn, will decrease the thermal resistance of the liquid film and thus increase heat transfer. Upward vapor flow has opposite effects

Presence of Noncondensable Gases in Condensers

- Experimental studies show that the presence of noncondensable gases in the vapor has a detrimental effect on condensation heat transfer

- Gas layer acts as a barrier between the vapor and the surface, and makes it difficult for the vapor to reach the surface and hence vapor has to diffuse first through noncondensable gas before reaching the surface

- Heat transfer in the presence of a noncondensable gas strongly depends on the nature of the vapor flow and the flow velocity



Correlations for Condensation heat transfer for other geometries

Film condensation inside Horizontal tubes :

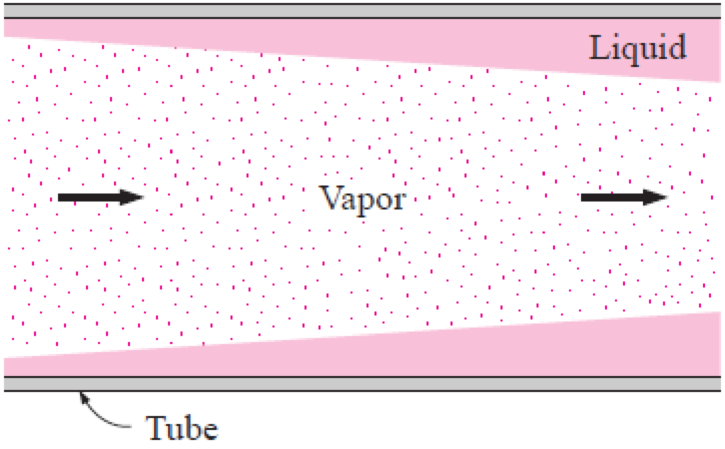
- Most condensation processes encountered in refrigeration and air-conditioning applications, however, involve condensation on the inner surfaces of horizontal or vertical tubes

- Heat transfer analysis of condensation inside tubes is complicated by the fact that it is strongly influenced by the vapor velocity and the rate of liquid accumulation on the walls of the tubes. **The numerical expression for low vapor velocities is**:

$$h_{\text{internal}} = 0.555\left[\frac{g\rho_l(\rho_l-\rho_v)k_l^3}{\mu_l(T_{\text{sat}}-T_s)}\left(h_{fg}+\frac{3}{8}C_{pl}(T_{\text{sat}}-T_s)\right)\right]^{1/4} \qquad \textbf{Eq.16}$$

- **Reynolds number of the vapor is to be evaluated** at the tube inlet conditions using the internal tube diameter as the characteristic length

$$\text{Re}_{\text{vapor}} = \left(\frac{\rho_v \mathcal{V}_v D}{\mu_v}\right)_{\text{inlet}} < 35{,}000$$

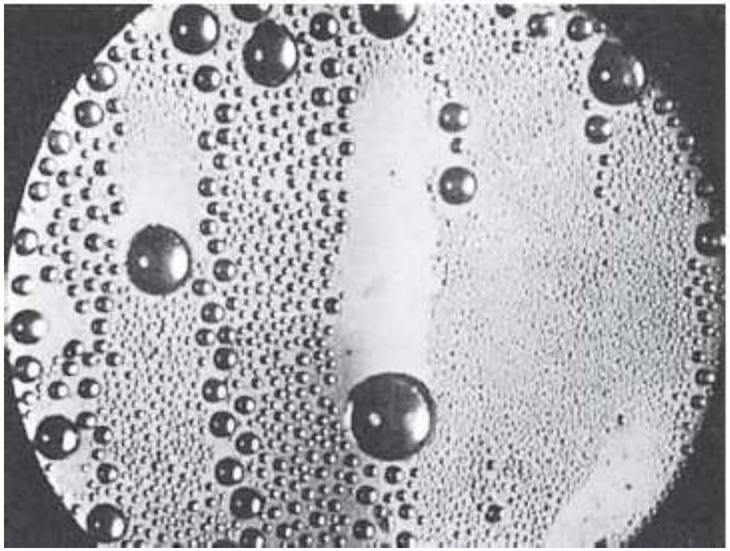


Dropwise Condensation

- Dropwise condensation, characterized by countless droplets of varying diameters on the condensing surface instead of a continuous liquid film, is one of the most effective mechanisms of heat transfer, and extremely large heat transfer coefficients can be achieved with this mechanism

- Small droplets that form at the nucleation sites on the surface grow as a result of continued condensation, coalesce into large droplets, and slide down when they reach a certain size, clearing the surface and exposing it to vapor

- There is no liquid film in this case to resist heat transfer. As a result, with dropwise condensation, heat transfer coefficients can be achieved that are more than 10 times larger than those associated with film condensation



Dropwise condensation of steam on a vertical surface

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Dropwise Condensation

- The challenge in dropwise condensation is not to achieve it, but rather, to sustain it for prolonged periods of time

- Dropwise condensation is achieved by adding a promoting chemical into the vapor, treating the surface with a promoter chemical, or coating the surface with a polymer such as Teflon or a noble metal

- Promoters used include various waxes and fatty acids such as oleic, stearic, and linoic acids

Heat transfer correlations for dropwise condensation of steam on copper surfaces:

$$h_{\text{dropwise}} = \begin{cases} 51{,}104 + 2044T_{\text{sat}}, & 22^\circ\text{C} < T_{\text{sat}} < 100^\circ\text{C} \\ 255{,}310 & T_{\text{sat}} > 100^\circ\text{C} \end{cases}$$

where T_{sat} is in °C and the heat transfer coefficient h_{dropwise} is in $\text{W/m}^2 \cdot {}^\circ\text{C}$

- High heat transfer coefficients achievable with dropwise condensation are of little significance if the material of the condensing surface is not a good conductor like copper or if the thermal resistance on the other side of the surface is too large

HEAT TRANSFER - 80329

MODULE V

HEAT EXCHANGERS & RADIATIVE HEAT TRANSFER

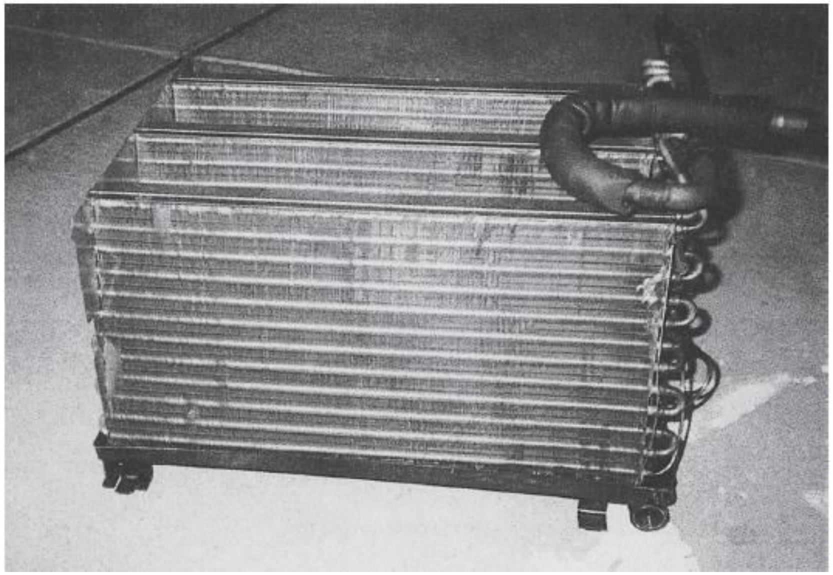
Heat exchangers – classification and overall heat transfer coefficient

Part 1

Dr. Satish Idury
Associate Professor

What is heat exchanger?

- Heat exchangers are devices that facilitate the exchange of heat between two fluids that are at different temperatures while keeping them from mixing with each other
- Heat exchangers are commonly used in practice in a wide range of applications, from heating and air-conditioning systems in a household, to chemical processing and power production in large plants
- Heat transfer in a heat exchanger usually involves convection in each fluid and conduction through the wall separating the two fluids
- Heat exchangers are manufactured in a variety of types due to practical application and therefore there are various classification of heat exchangers



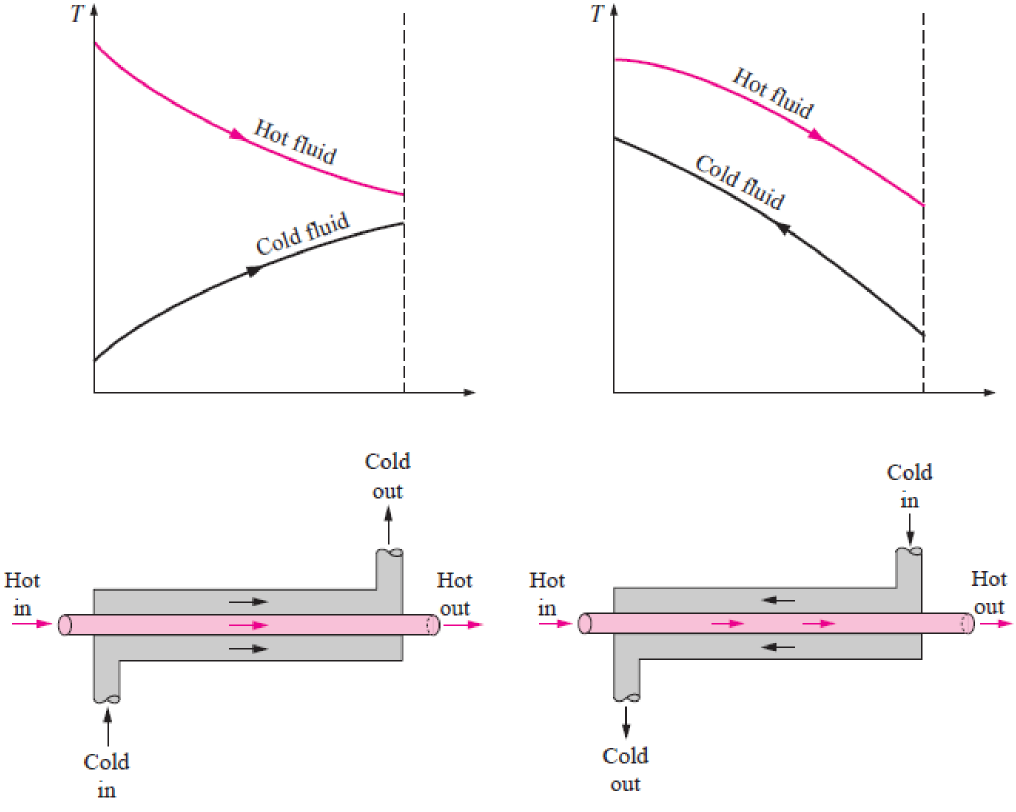
A gas-to-liquid compact heat exchanger for a residential air-conditioning system

Classification of heat exchangers

- The simplest type of heat exchanger consists of two concentric pipes of different diameters, as shown in Fig. called the **double-pipe** heat exchanger

- Two types of flow arrangement are possible in a double-pipe heat exchanger. In **parallel flow,** both the hot and cold fluids enter the heat exchanger at the same end and move in the *same* direction

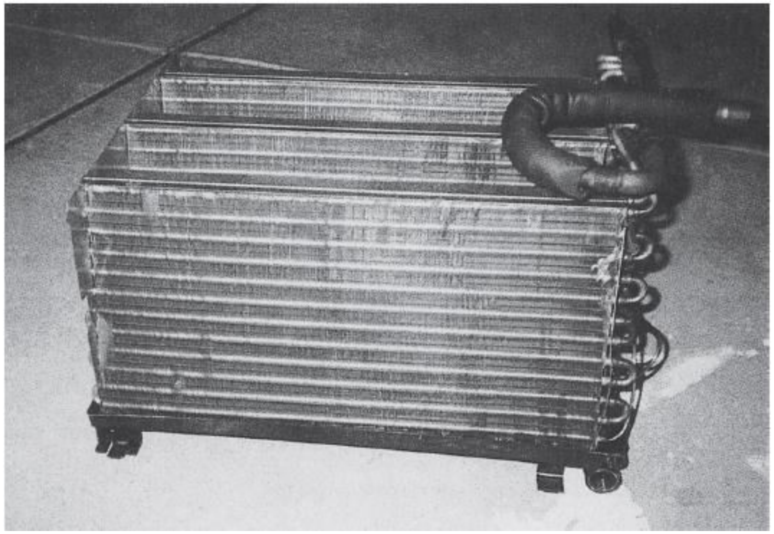
- In **counter flow,** the hot and cold fluids enter the heat exchanger at opposite ends and flow in opposite directions



(*a*) Parallel flow (*b*) Counter flow

Classification of heat exchangers

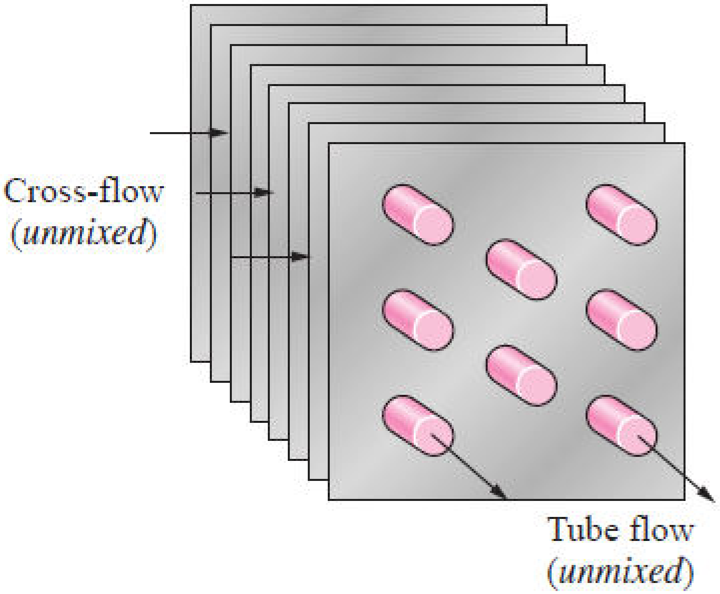
- Heat exchanger, which is specifically designed to realize a large heat transfer surface area per unit volume, is the **compact** heat exchanger
- The large surface area in compact heat exchangers is obtained by attaching closely spaced thin plate or corrugated fins to the walls separating the two fluids
- Compact heat exchangers are commonly used in gas-to-gas and gas-to liquid (or liquid-to-gas) heat exchangers to counteract the low heat transfer coefficient associated with gas flow with increased surface area
- The ratio of the heat transfer surface area of a heat exchanger to its volume is called the area density . A heat exchanger with 700 m^2/m^3 is classified as being compact



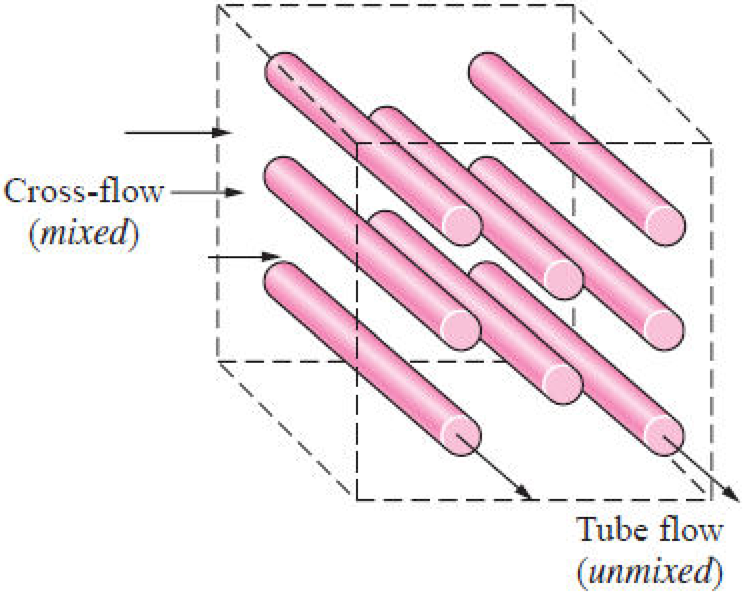
A gas-to-liquid compact heat exchanger for a residential air-conditioning system

Classification of heat exchangers

- In compact heat exchangers, the two fluids usually move perpendicular to each other, and such flow configuration is called **cross-flow.** The cross-flow is further classified as unmixed and mixed flow, depending on the flow configuration
- In (*a*) the cross-flow is said to be *unmixed* since the plate fins force the fluid to flow through a particular interfin spacing and prevent it from moving in the transverse direction (i.e., parallel to the tubes). The cross-flow in (*b*) is said to be *mixed* since the fluid now is free to move in the transverse direction
- The presence of mixing in the fluid can have a significant effect on the heat transfer characteristics of the heat exchanger

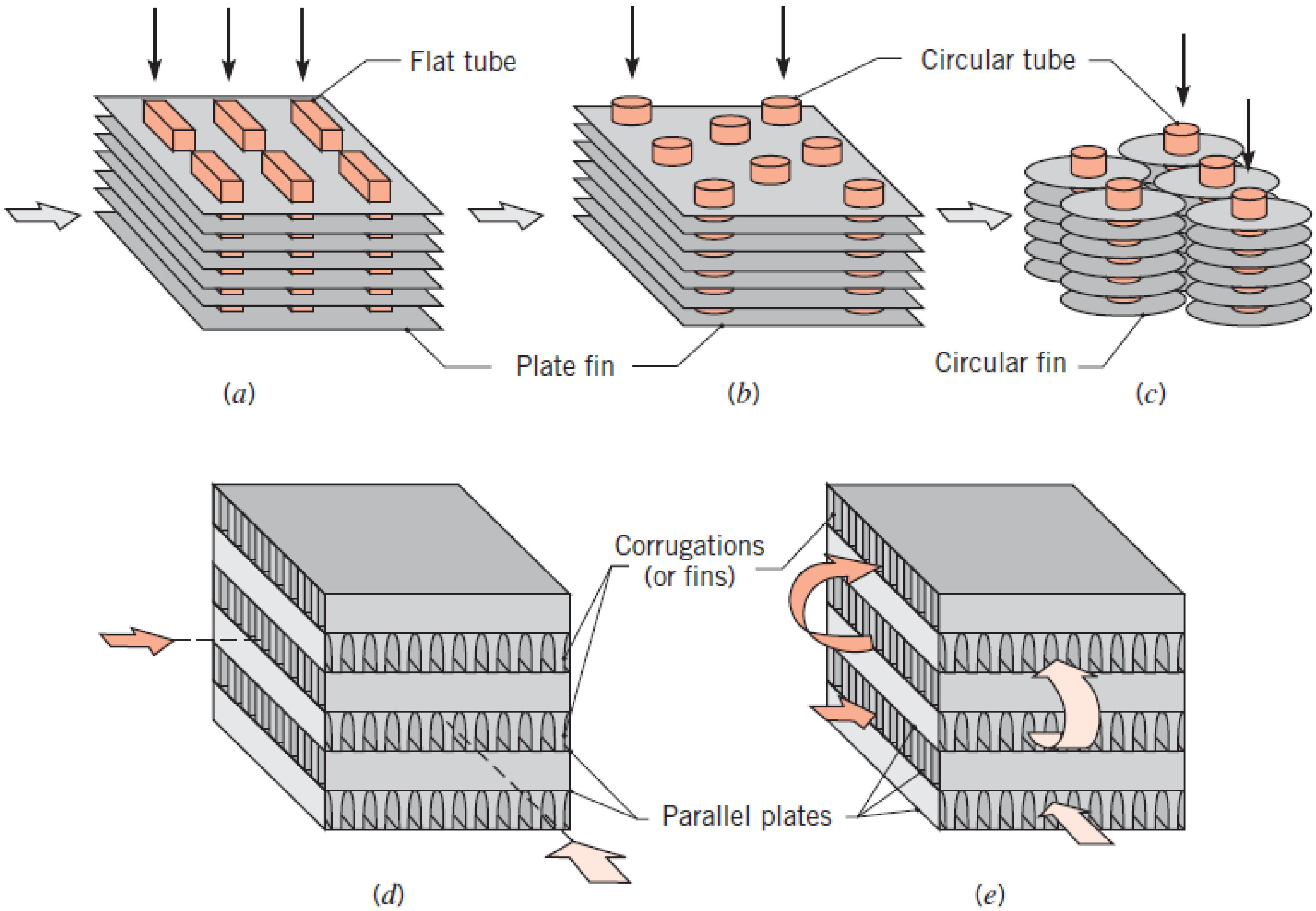


(*a*) Both fluids unmixed



(*b*) One fluid mixed, one fluid unmixed

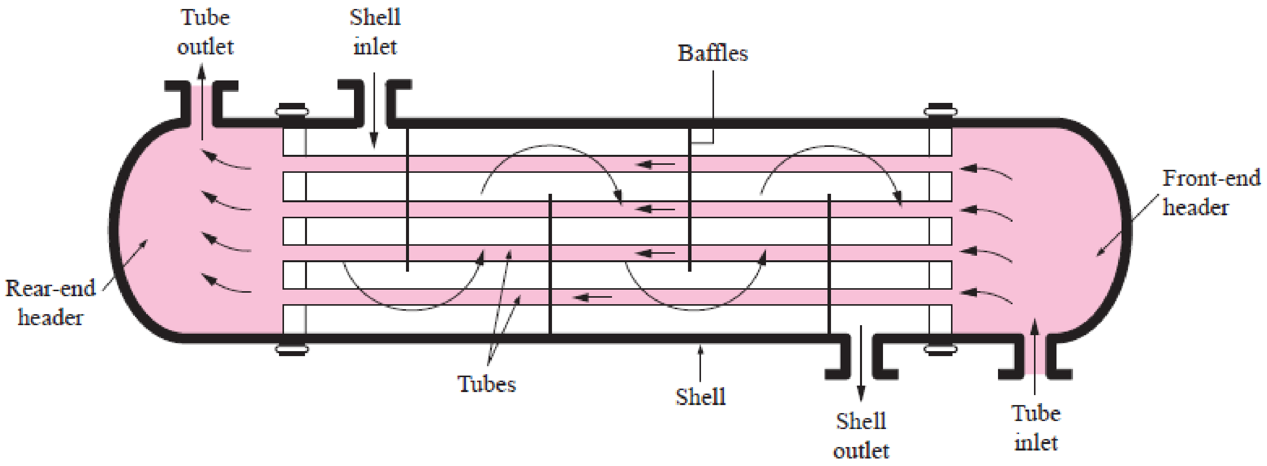
Compact heat exchanger configurations

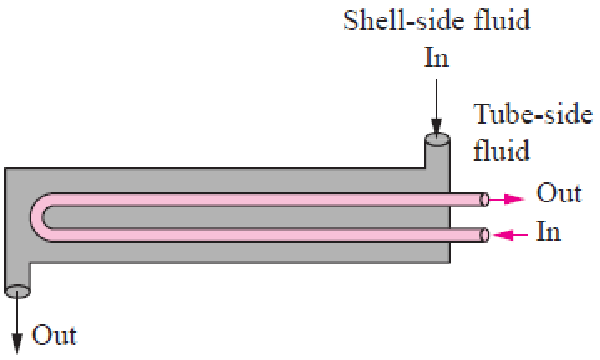


Compact heat exchanger cores. (*a*) Fin–tube (flat tubes, continuous plate fins). (*b*) Fin–tube (circular tubes, continuous plate fins). (*c*) Fin–tube (circular tubes, circular fins). (*d*) Plate–fin (single pass). (*e*) Plate–fin (multipass)

Shell and tube heat exchanger

- The most common type of heat exchanger in industrial applications is the **shell-and-tube** heat exchanger, shown in Fig. Shell-and-tube heat exchangers contain a large number of tubes (sometimes several hundred) packed in a shell with their axes parallel to that of the shell
- *Baffles* are commonly placed in the shell to force the shell-side fluid to flow across the shell to enhance heat transfer and to maintain uniform spacing between the tube
- Shell and tube heat exchangers are not suitable for use in automotive and aircraft applications because of their relatively large size and weight
- Shell-and-tube heat exchanger open to some large flow areas called *headers* at both ends of the shell, where the tube-side fluid accumulates before entering the tubes and after leaving them

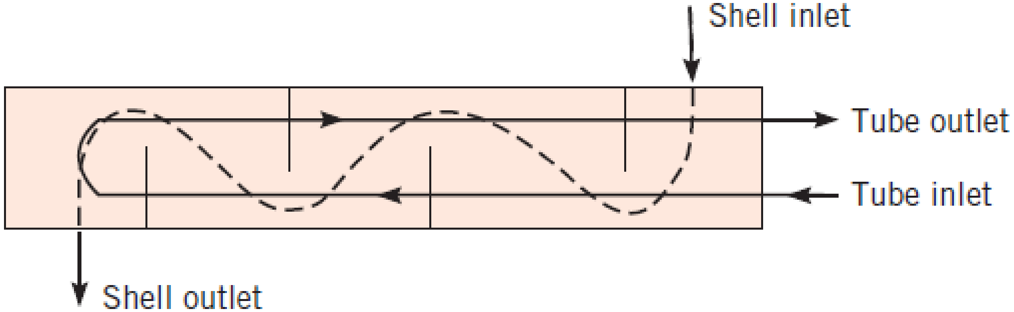




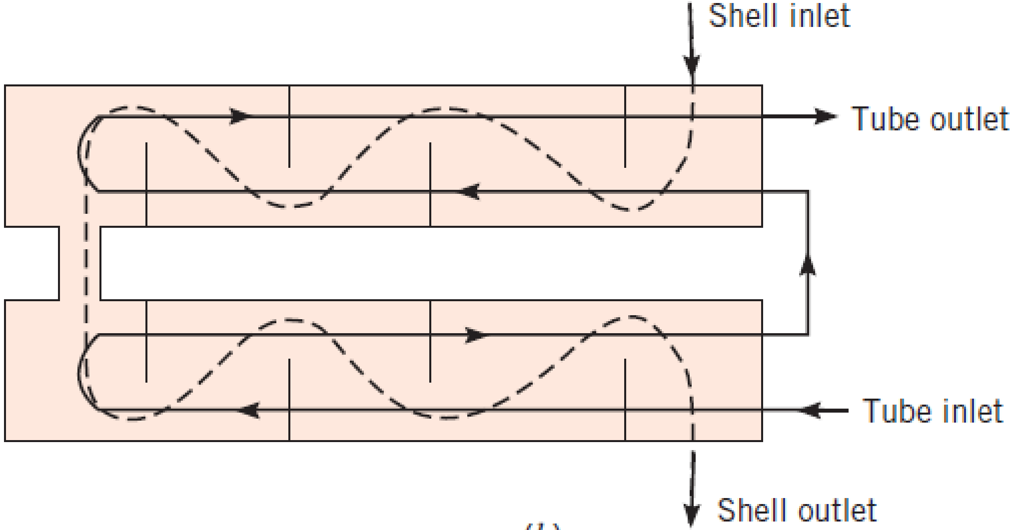
(*a*) One-shell pass and two-tube passes

Shell and tube heat exchanger

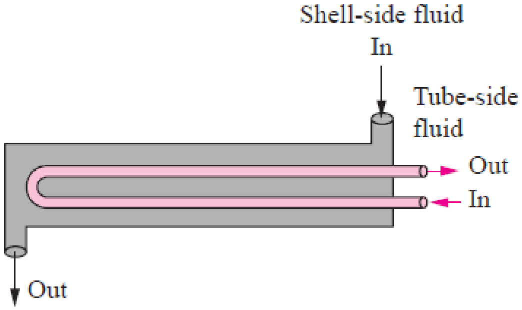
- Heat exchangers in which all the tubes make one U-turn in the shell, for example, are called *one-shell-pass and two tube- passes* heat exchangers
- Heat exchanger that involves two passes in the shell and four passes in the tubes is called a *two-shell-passes and four-tube-passes* heat exchanger



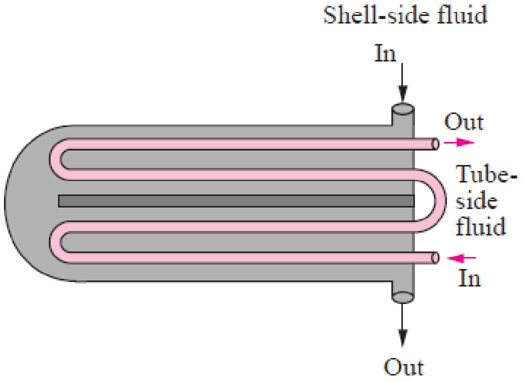
(*a*)



(*b*)



(*a*) One-shell pass and two-tube passes



(*b*) Two-shell passes and four-tube passes

Overall heat transfer coefficient

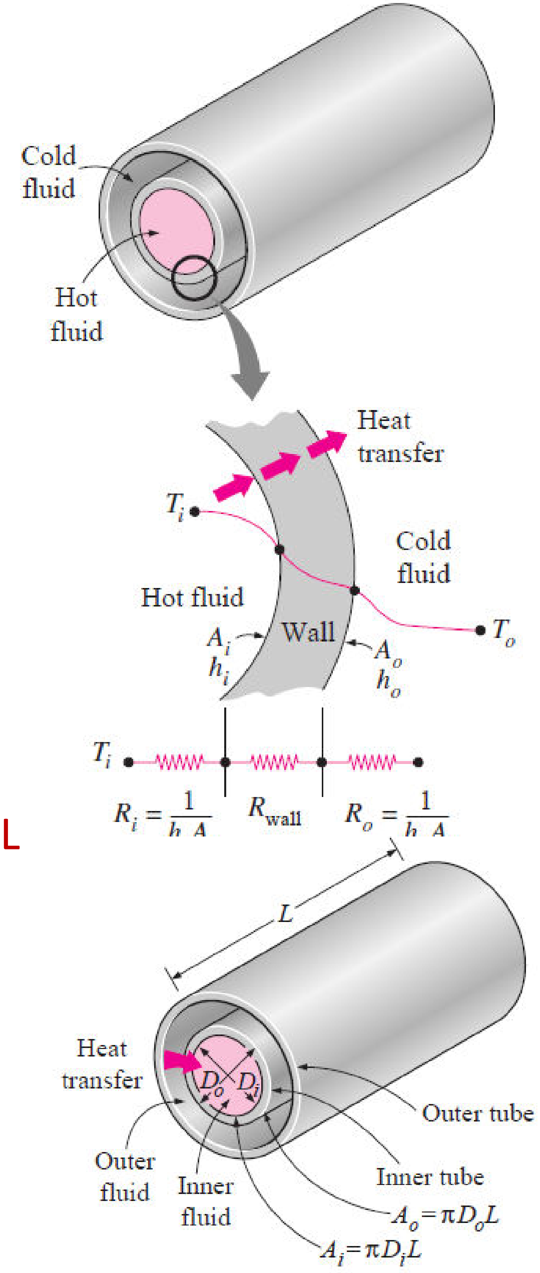
- In a heat exchanger, heat is first transferred from the hot fluid to the wall by convection, through the wall by conduction, and from the wall to the cold fluid again by convection. Any radiation effects are usually included in the convection heat transfer coefficients
- As shown in Fig. the subscripts *i* and *o* represent the inner and outer surfaces of the heat exchanger.
- For a double-pipe heat exchanger, we have $A_i = \pi D_i L$ and $A_o = \pi D_o L$, and the *thermal resistance* of the wall:

$$R_{\text{wall}} = \frac{\ln(D_o/D_i)}{2\pi k L}$$

where k is the thermal conductivity of the wall material and L is the length of the tube. Then the total thermal resistance

$$R = R_{\text{total}} = R_i + R_{\text{wall}} + R_o = \frac{1}{h_i A_i} + \frac{\ln(D_o/D_i)}{2\pi k L} + \frac{1}{h_o A_o}$$

$$\dot{Q} = \frac{\Delta T}{R} = UA\,\Delta T = U_i A_i\,\Delta T = U_o A_o\,\Delta T$$



Overall heat transfer coefficient

- It is convenient to combine all the thermal resistances in the path of heat flow from the hot fluid to the cold one into a single resistance *R*

$$\dot{Q} = \frac{\Delta T}{R} = UA\,\Delta T = U_iA_i\,\Delta T = U_oA_o\,\Delta T$$

where *U* is the **overall heat transfer coefficient,** whose unit is W/m^2 · °C, which is identical to the unit of the ordinary convection coefficient *h*

$$\frac{1}{UA_s} = \frac{1}{U_iA_i} = \frac{1}{U_oA_o} = R = \frac{1}{h_iA_i} + R_{\text{wall}} + \frac{1}{h_oA_o}$$

$$U_iA_i = U_oA_o, \text{ but } U_i \neq U_o \text{ unless } A_i = A_o$$

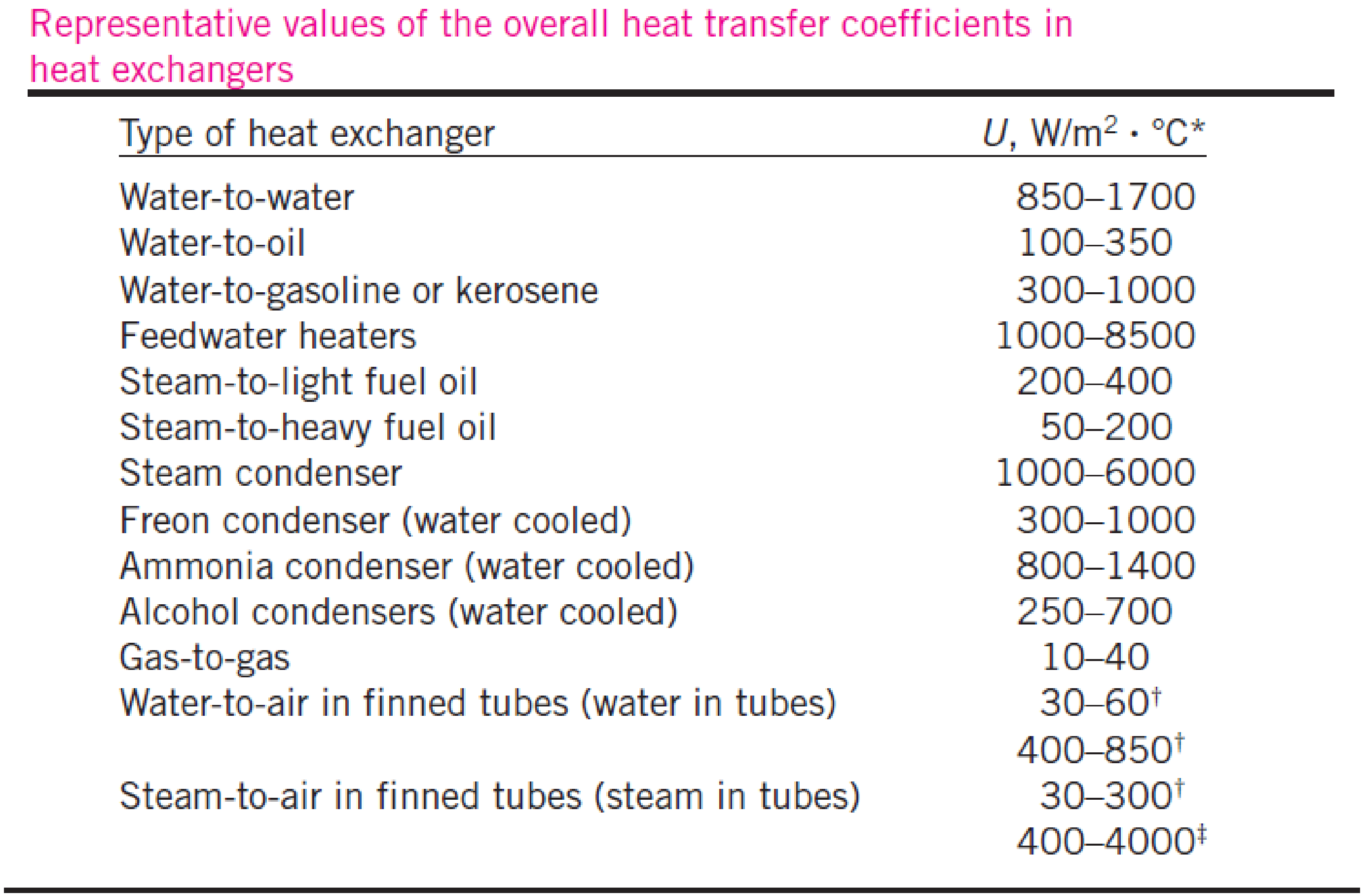
- Overall heat transfer coefficient *U* of a heat exchanger is meaningless unless the area on which it is based is specified

- When the wall thickness of the tube is small and the thermal conductivity of the tube material is high, as is usually the case, the thermal resistance of the tube is negligible (R_{wall} ~~ 0) and the inner and outer surfaces of the tube are almost identical ($A_i = A_o = A_s$)

Overall heat transfer coefficient

$$\frac{1}{U} \approx \frac{1}{h_i} + \frac{1}{h_o} \qquad \text{where } U = U_i = U_o$$

When one of the convection coefficients is *much smaller* than the other (say, $h_i << h_o$), we have $1/h_i >> 1/h_o$, and thus $U \sim\sim h_i$. Therefore, the smaller heat transfer coefficient creates a *bottleneck* on the path of heat flow and seriously impedes heat transfer



Representative values of the overall heat transfer coefficients in heat exchangers

Type of heat exchanger	U, W/m^2 · °C[*]
Water-to-water	850–1700
Water-to-oil	100–350
Water-to-gasoline or kerosene	300–1000
Feedwater heaters	1000–8500
Steam-to-light fuel oil	200–400
Steam-to-heavy fuel oil	50–200
Steam condenser	1000–6000
Freon condenser (water cooled)	300–1000
Ammonia condenser (water cooled)	800–1400
Alcohol condensers (water cooled)	250–700
Gas-to-gas	10–40
Water-to-air in finned tubes (water in tubes)	30–60[†]
	400–850[†]
Steam-to-air in finned tubes (steam in tubes)	30–300[†]
	400–4000[‡]

Overall heat transfer coefficient

- When the tube is *finned* on one side to enhance heat transfer, the total heat transfer surface area on the finned side becomes

$$A_s = A_{\text{total}} = A_{\text{fin}} + A_{\text{unfinned}}$$

- For short fins of high thermal conductivity, we can use this total area in the convection resistance relation $R_{\text{conv}} = 1/hA_s$ since the fins in this case will be very nearly isothermal

- If the fin is not isothermal, the effective surface area can be determined by :

$$A_s = A_{\text{unfinned}} + \eta_{\text{fin}} A_{\text{fin}}$$

where η_{fin} *is the fin efficiency. This way, the temperature drop along the fins is accounted for. Note that* $\eta_{fin} = 1$ *for isothermal fins*

Fouling in heat exchangers

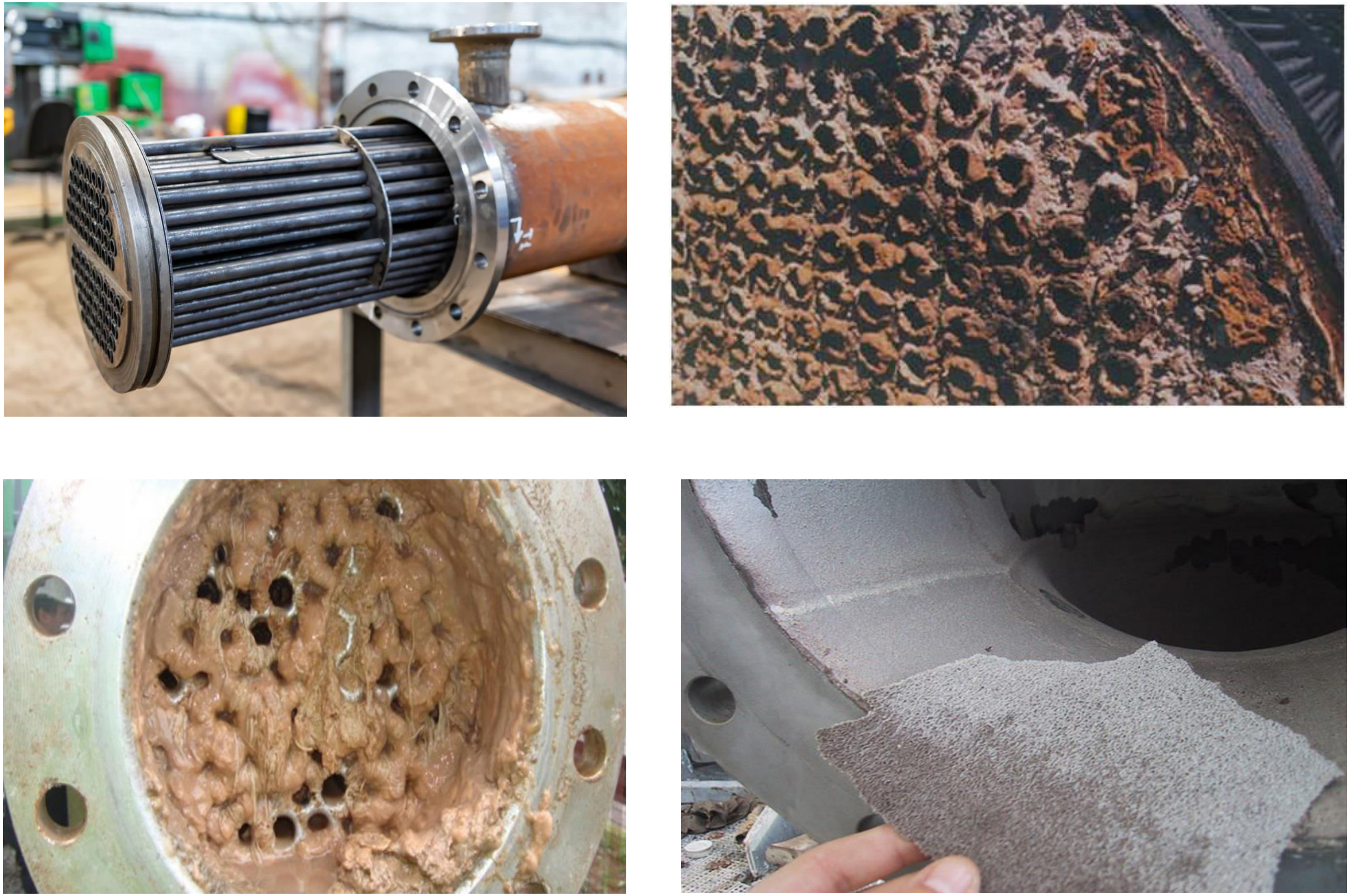


Image courtesy : Google

Fouling factor

- The performance of heat exchangers usually deteriorates with time as a result of accumulation of *deposits* on heat transfer surfaces.

- The layer of deposits represents *additional resistance* to heat transfer and causes the rate of heat transfer in a heat exchanger to decrease

- The net effect of these accumulations on heat transfer is represented by a **fouling factor** R_f , which is a measure of the *thermal resistance* introduced by fouling

Fouling in heat exchangers occurs by following factors:

- Precipitation of solid deposits in a fluid on heat transfer surfaces
- Corrosion and other chemical fouling
- Biological fouling

- The fouling factor is obviously zero for a new heat exchanger and increases with time as the solid deposits build up on the heat exchanger surface

Fouling increases with increasing temperature and decreasing velocity

Fouling factor

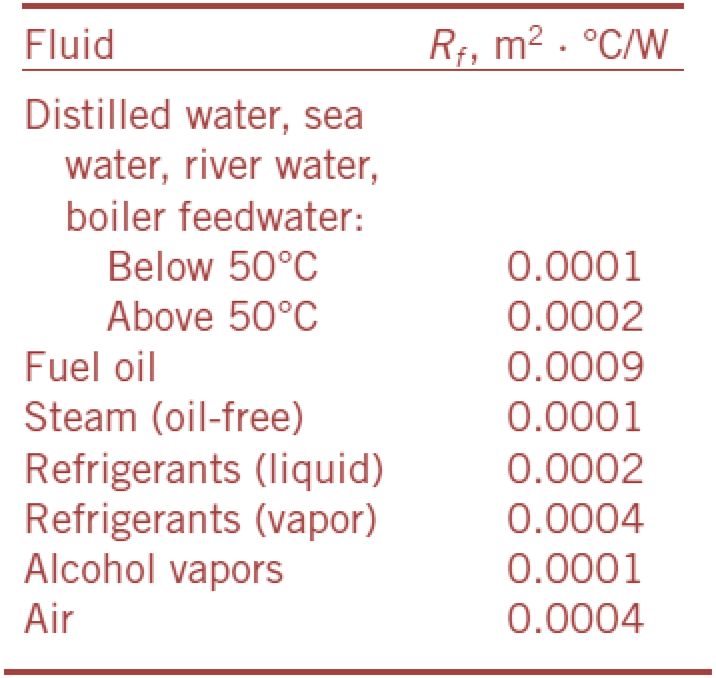
For an unfinned shell-and-tube heat exchanger the overall heat transfer coefficient expression gets modified by considering internal and external fouling factors:

$$\frac{1}{UA_s} = \frac{1}{U_i A_i} = \frac{1}{U_o A_o} = R = \frac{1}{h_i A_i} + \frac{R_{f,i}}{A_i} + \frac{\ln(D_o/D_i)}{2\pi k L} + \frac{R_{f,o}}{A_o} + \frac{1}{h_o A_o}$$

With inclusion of surface fouling and fin (extended surface) effects, the overall heat transfer coefficient is further modified as below:

$$\frac{1}{UA} = \frac{1}{(\eta_o hA)_c} + \frac{R''_{f,c}}{(\eta_o A)_c} + R_w + \frac{R''_{f,h}}{(\eta_o A)_h} + \frac{1}{(\eta_o hA)_h}$$

In the above expression c and h refers to cold and hot side of heat exchanger respectively and η_o is the overall surface efficiency or temperature effectiveness of a finned surface



Fluid	R_f, m$^2\cdot$°C/W
Distilled water, sea water, river water, boiler feedwater:	
Below 50°C	0.0001
Above 50°C	0.0002
Fuel oil	0.0009
Steam (oil-free)	0.0001
Refrigerants (liquid)	0.0002
Refrigerants (vapor)	0.0004
Alcohol vapors	0.0001
Air	0.0004

Representative fouling factors (thermal resistance due to fouling for a unit surface area)

Fouling factor

$$\frac{1}{UA} = \frac{1}{(\eta_o hA)_c} + \frac{R''_{f,c}}{(\eta_o A)_c} + R_w + \frac{R''_{f,h}}{(\eta_o A)_h} + \frac{1}{(\eta_o hA)_h}$$

The quantity η_o in the above equation is the overall surface efficiency or temperature effectiveness of a finned surface. It is defined such that, for the hot or cold surface without fouling, the heat transfer rate is

$$q = \eta_o hA(T_b - T_\infty)$$

h should be replaced by below expression if fouling is considered:

$$U_p = h/(1 + hR''_f).$$

where T_b is the base surface temperature and A is the total (fin plus exposed base) surface area

$$\eta_o = 1 - \frac{A_f}{A}(1 - \eta_f)$$

where A_f is the entire fin surface area and η_f is the efficiency of a single fin. the ratio of fin surface area to the total surface area has been expressed as A_f/A

If a straight or pin fin of length L is used and an adiabatic tip is assumed:

$$\eta_f = \frac{\tanh(mL)}{mL}$$

where $m = (2h/kt)^{1/2}$ and t is the fin thickness

Analysis of heat exchangers – general assumptions

- An engineer often lands in a position to *select a heat exchanger* that will achieve a *specified temperature change* in a fluid stream of known mass flow rate, or to *predict the outlet temperatures* of the hot and cold fluid streams in a *specified heat exchanger*
- *Log mean temperature difference* (or LMTD) method is best suited for the first task viz. quantifying temperature changes in a fluid stream of known mass flow rate
- *Effectiveness–NTU* method is suited for predicting outlet temperatures of hot and cold fluid streams
- Heat exchangers are steady flow devices, hence the mass flow rate of each fluid remains constant, and the fluid properties such as temperature and velocity at any inlet or outlet remain the same
- The fluid streams experience little or no change in their velocities and elevations, and thus the *kinetic* and *potential energy changes* are negligible. Within a particular temperature range, specific heat is regarded constant
- *Axial heat conduction* along the tube is usually insignificant and can be considered negligible and heat exchanger is insulated

Analysis of heat exchangers – general assumptions

- The *first law of thermodynamics* requires that the rate of heat transfer from the hot fluid be equal to the rate of heat transfer to the cold one:

$$\dot{Q} = \dot{m}_c C_{pc}(T_{c,\text{ out}} - T_{c,\text{ in}})$$

$$\dot{Q} = \dot{m}_h C_{ph}(T_{h,\text{ in}} - T_{h,\text{ out}})$$

$\dot{m}_c, \dot{m}_h$ = mass flow rates

C_{pc}, C_{ph} = specific heats

$T_{c,\text{ out}}, T_{h,\text{ out}}$ = outlet temperatures

$T_{c,\text{ in}}, T_{h,\text{ in}}$ = inlet temperatures

where the subscripts *c* and *h* stand for *cold* and *hot* fluids, respectively

- Heat transfer rate $\dot{Q}$ is taken to be a positive quantity, and its direction is understood to be from the hot fluid to the cold one in accordance with the second law of thermodynamics
- The product of the *mass flow rate* and the *specific heat* of a fluid is called heat capacity

$$C_h = \dot{m}_h C_{ph} \quad \text{and} \quad C_c = \dot{m}_c C_{pc}$$

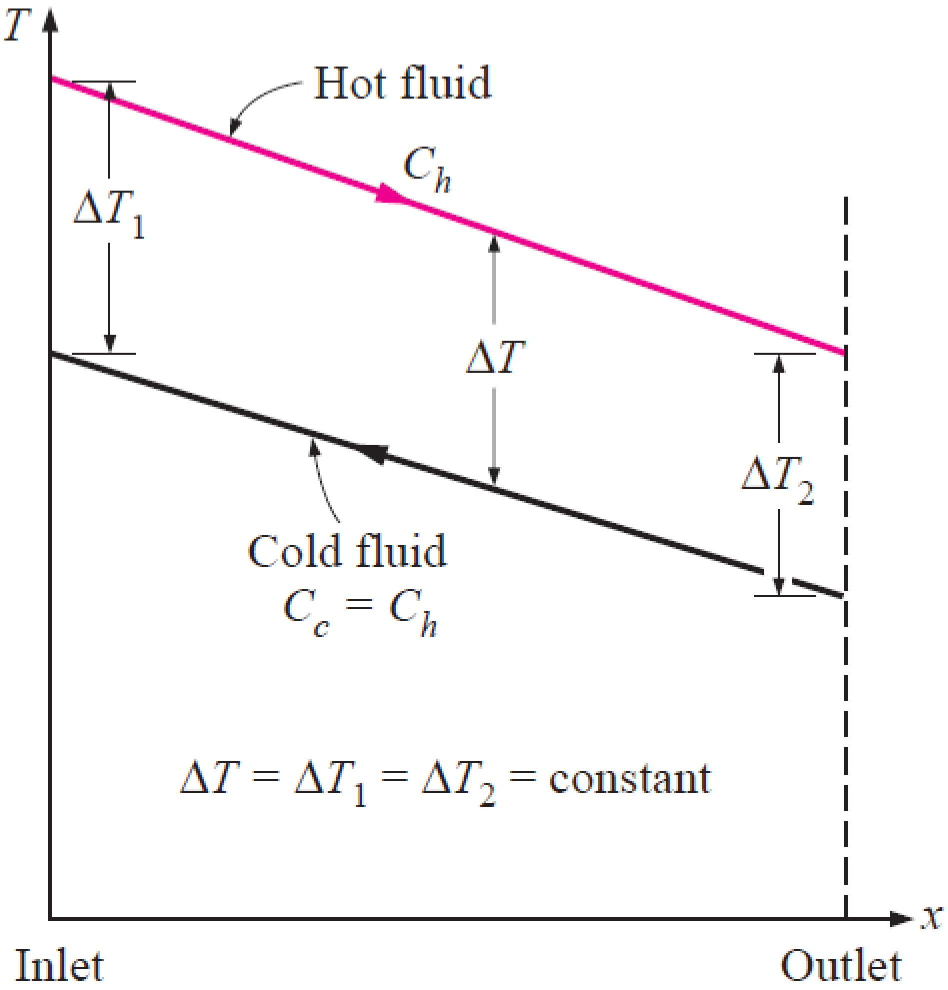
In a heat exchanger, the fluid with a large heat capacity rate will experience a small temperature change, and the fluid with a small heat capacity rate will experience a large temperature change. Doubling the mass flow rate of a fluid while leaving everything else unchanged will halve the temperature change of that fluid

Analysis of heat exchangers – general assumptions

$$\dot{Q} = C_c(T_{c,\,\text{out}} - T_{c,\,\text{in}})$$

$$\dot{Q} = C_h(T_{h,\,\text{in}} - T_{h,\,\text{out}})$$

The only time the temperature rise of a cold fluid is equal to the temperature drop of the hot fluid is when the heat capacity rates of the two fluids are equal to each other



Analysis of heat exchangers – general assumptions

- One of the fluids in a condenser or a boiler undergoes a phase-change process, and the rate of heat transfer is expressed as:

$$\dot{Q} = \dot{m} h_{fg}$$

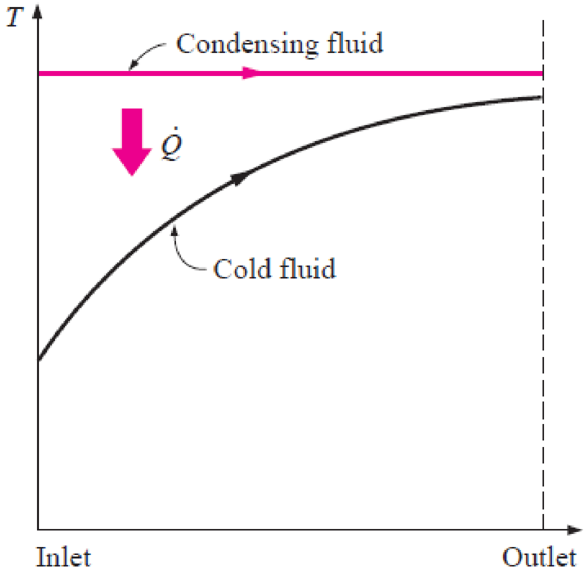
where $\dot{m}$ is the rate of evaporation or condensation of the fluid and h_{fg} is the enthalpy of vaporization of the fluid at the specified temperature or pressure

C= $\dot{m}$ C_p →∞ when T → 0, so that the heat transfer rate Q = $\dot{m}$ C_p ΔT is a finite quantity during phase change process

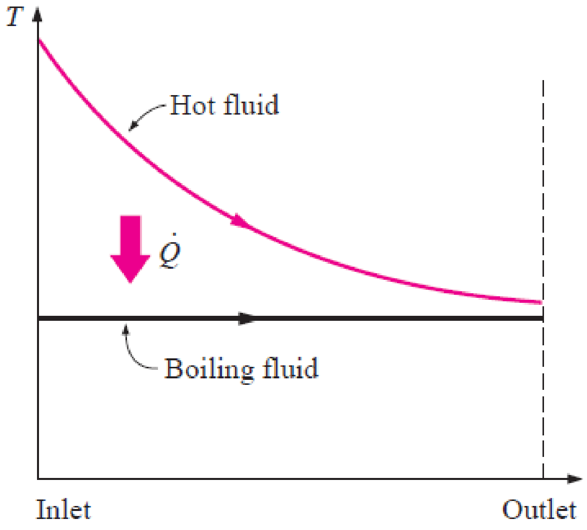
- The rate of heat transfer in a heat exchanger can also be expressed in an analogous manner to Newton's law of cooling as:

$$\dot{Q} = UA_s \Delta T_m$$

where U is the overall heat transfer coefficient, A_s is the heat transfer area, and ΔT_m is an appropriate average temperature difference between the two fluids



(a) Condenser ($C_h \to \infty$)



(b) Boiler ($C_c \to \infty$)

HEAT TRANSFER - 80329

MODULE V

HEAT EXCHANGERS & RADIATIVE HEAT TRANSFER

MALLA REDDY ENGINEERING COLLEGE

MREC

Heat exchangers – LMTD and NTU methods

Part 2

Dr. Satish Idury
Associate Professor

Log Mean Temperature Difference method (LMTD)

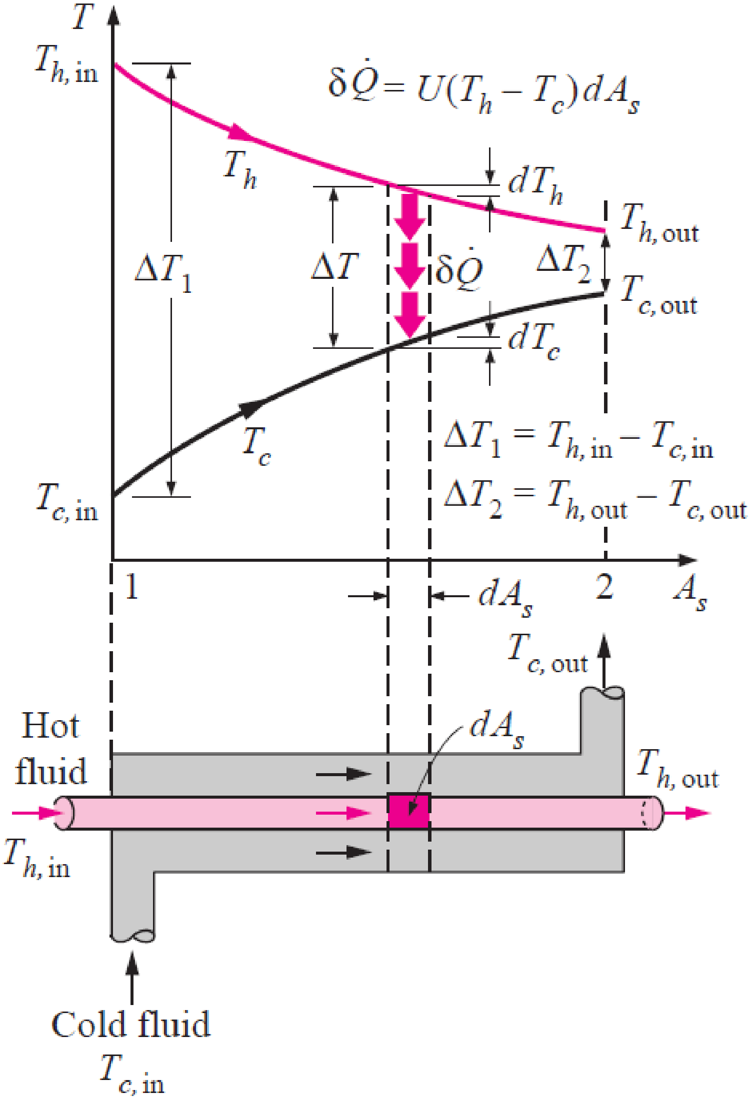
- The rate of heat transfer in a heat exchanger can be expressed in a manner that is analogous to Newton's law of cooling

$$\dot{Q} = UA_s\,\Delta T_m$$

- In order to develop a relation for the equivalent average temperature difference (ΔT_m) between the two fluids, consider the *parallel-flow double-pipe* heat exchanger shown

- The temperature difference ΔT between the hot and cold fluids is large at the inlet of the heat exchanger but decreases exponentially toward the outlet

- Assuming the outer surface of heat exchanger to be insulated and disregarding changes in potential and kinetic energies, an energy balance on fluid for a differential section of heat exchanger can be expressed



Log Mean Temperature Difference method (LMTD)

- An energy balance on each fluid in a differential section of the heat exchanger is:

$$\delta\dot{Q} = -\dot{m}_h C_{ph}\, dT_h \qquad \delta\dot{Q} = \dot{m}_c C_{pc}\, dT_c$$

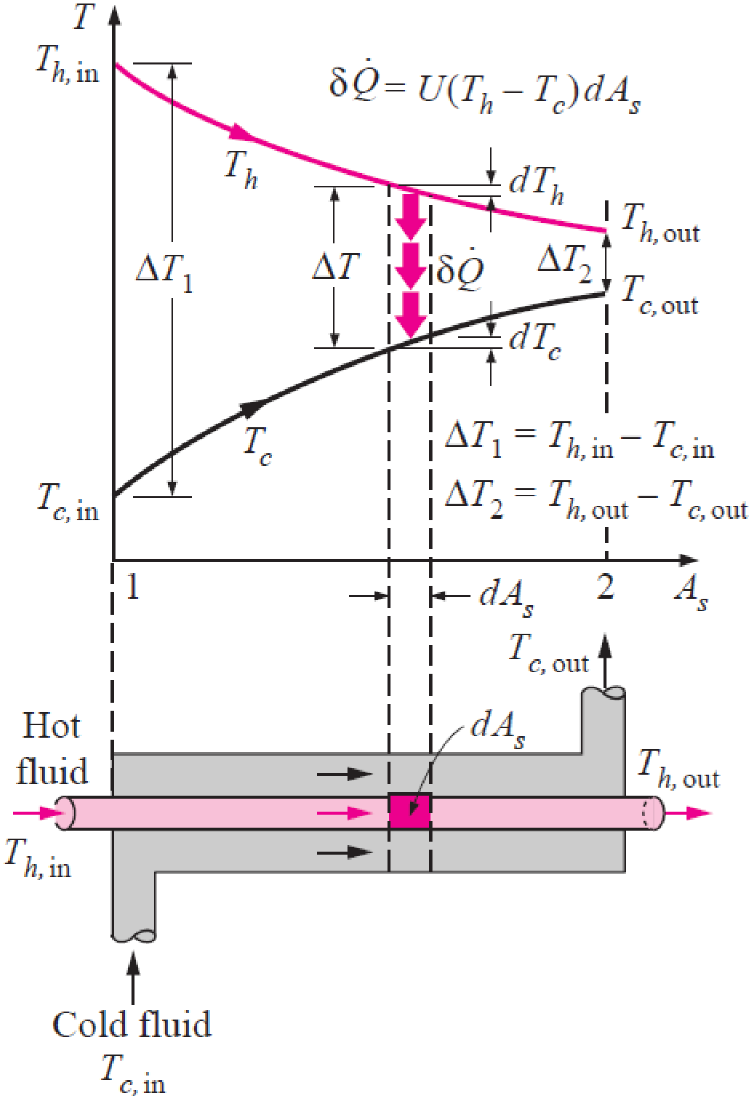
- The rate of heat loss from the hot fluid at any section of a heat exchanger is equal to the rate of heat gain by the cold fluid in that section

$$dT_h = -\frac{\delta\dot{Q}}{\dot{m}_h C_{ph}} \qquad dT_c = \frac{\delta\dot{Q}}{\dot{m}_c C_{pc}}$$

$$dT_h - dT_c = d(T_h - T_c) = -\delta\dot{Q}\left(\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_c C_{pc}}\right)$$

- The rate of heat transfer in the differential section of the heat exchanger is:

$$\delta\dot{Q} = U(T_h - T_c)\, dA_s$$



Log Mean Temperature Difference method (LMTD)

$$dT_h - dT_c = d(T_h - T_c) = -\delta\dot{Q}\left(\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_c C_{pc}}\right)$$ **Eq.1**

- The rate of heat transfer in the differential section of the heat exchanger is:

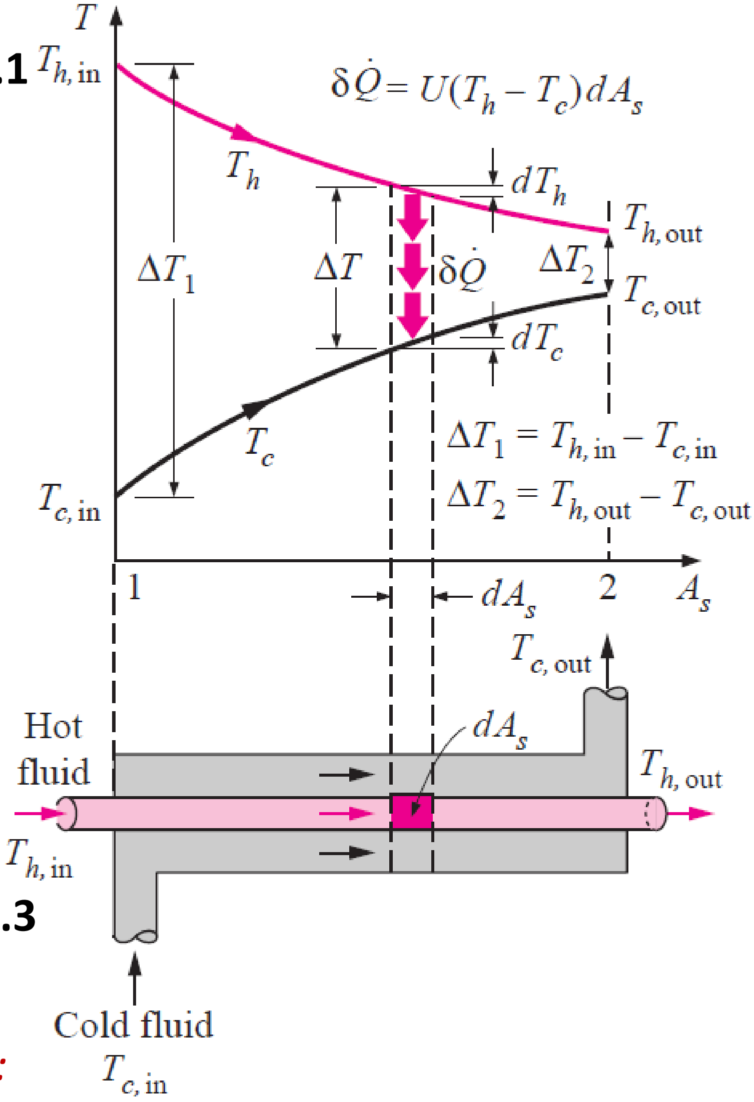
$$\delta\dot{Q} = U(T_h - T_c)\, dA_s$$ **Eq.2**

On substitution of Eq.2 in Eq.1

$$\frac{d(T_h - T_c)}{T_h - T_c} = -U\, dA_s\left(\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_c C_{pc}}\right)$$

- Integrating from the inlet of the heat exchanger to its outlet

$$\ln\frac{T_{h,\,\mathrm{out}} - T_{c,\,\mathrm{out}}}{T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}}} = -UA_s\left(\frac{1}{\dot{m}_h C_{ph}} + \frac{1}{\dot{m}_c C_{pc}}\right)$$ **Eq.3**



But from first law of thermodynamics it is known that:

$$\dot{Q} = \dot{m}_h C_{ph}(T_{h,\,\mathrm{in}} - T_{h,\,\mathrm{out}}) \qquad \dot{Q} = \dot{m}_c C_{pc}(T_{c,\,\mathrm{out}} - T_{c,\,\mathrm{in}})$$ **Eq.4**

Contd---

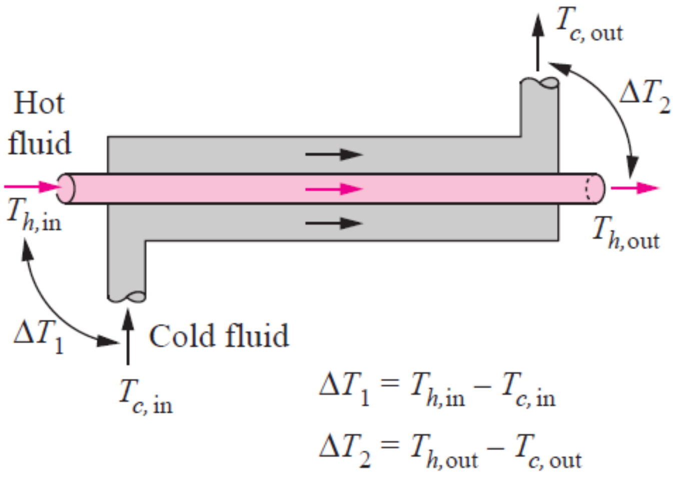
Log Mean Temperature Difference method (LMTD)

On substitution of Eq.4 in right hand side of Eq.3 and further simplification gives

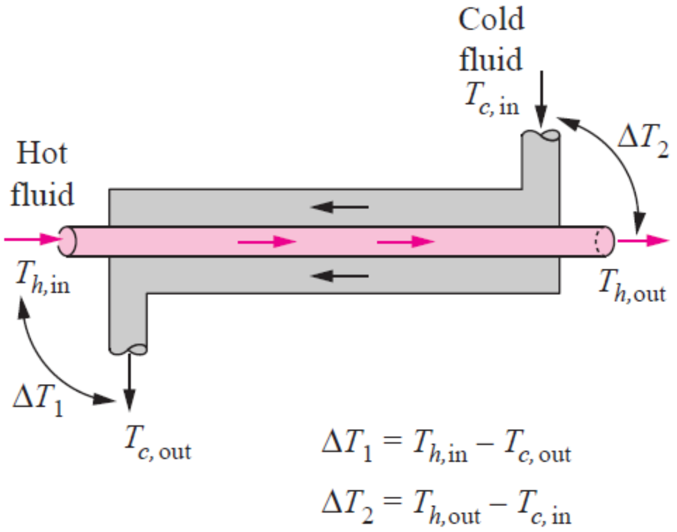
$$\dot{Q} = UA_s \,\Delta T_{\text{lm}}$$

where ΔT_{lm} is given by: $$\Delta T_{\text{lm}} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

- ΔT_{lm} is the **log mean temperature difference,** which is the suitable form of the average temperature difference for use in the analysis of heat exchangers
- ΔT_1 and ΔT_2 represents the temperature difference between two fluids at inlet and outlet of a heat exchanger
- ΔT_{lm} is obtained by tracing the actual temperature profile of the fluids along the heat exchanger and is an exact representation of the average temperature difference between the hot and cold fluids. ΔT_{lm} truly represents the exponential decay of local temperature difference
- $\Delta T_{lm} < \Delta T_{am}$. Therefore ΔT_{am} will overestimate the range of heat transfer



(*a*) Parallel-flow heat exchangers

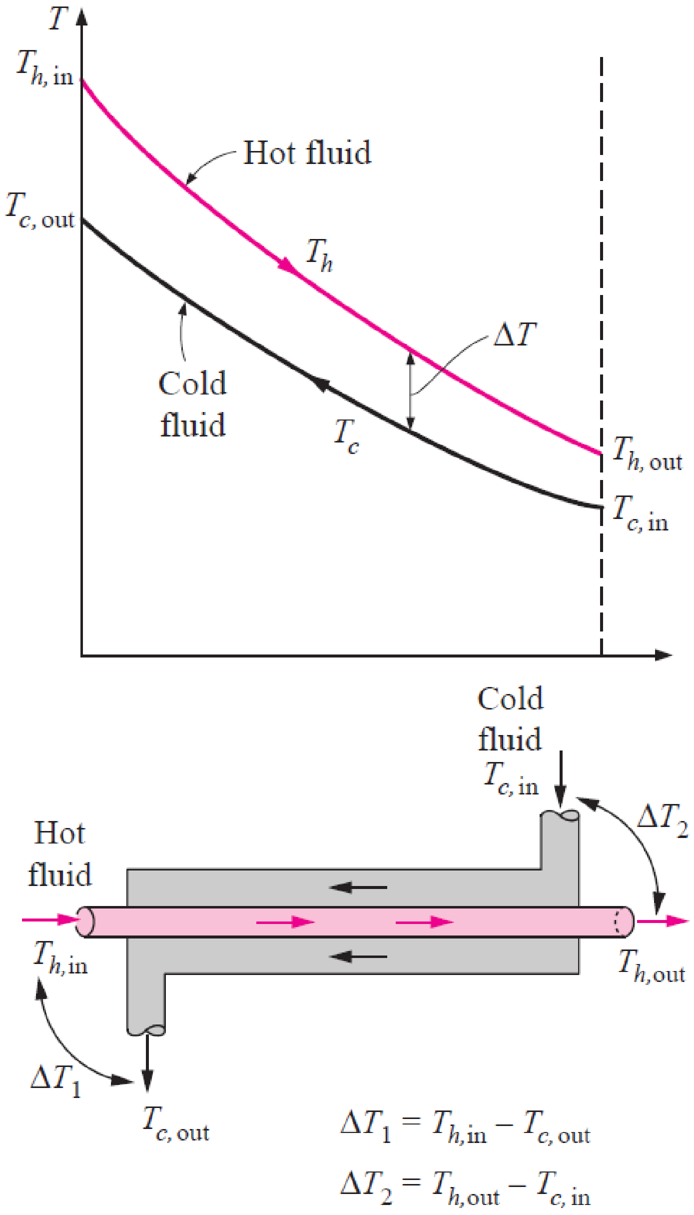


(*b*) Counter-flow heat exchangers

LMTD – Counter flow heat exchangers

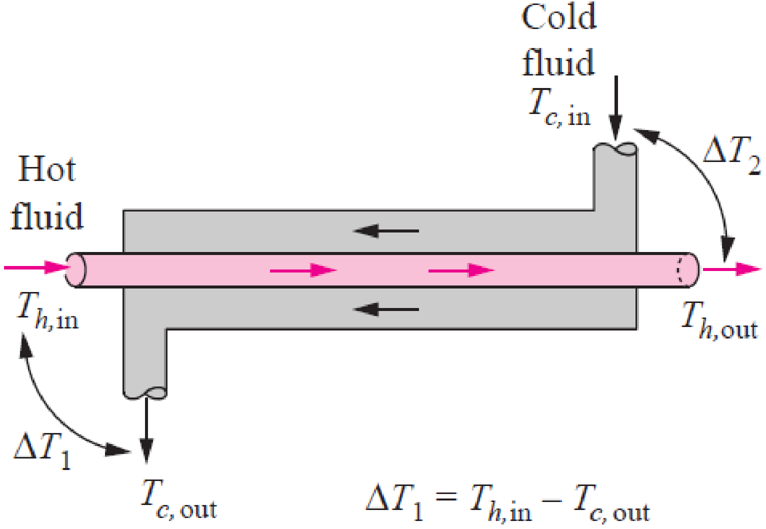
- Hot and cold fluids enter the heat exchanger from opposite ends, and the outlet temperature of the *cold fluid* in this case may exceed the outlet temperature of the *hot fluid*
- In the limiting case, the cold fluid will be heated to the inlet temperature of the hot fluid. However, the outlet temperature of the cold fluid can *never* exceed the inlet temperature of the hot fluid, since this would be a violation of the second law of thermodynamics.
- The relation above for the LMTD developed using a parallel-flow heat exchange is equally applicable for counter flow heat exchanger
- ΔT_1 and ΔT_2 are expressed as shown below:

$$\Delta T_1 = T_{h,\text{in}} - T_{c,\text{out}}$$
$$\Delta T_2 = T_{h,\text{out}} - T_{c,\text{in}}$$



LMTD – Counter flow heat exchangers

- For specified inlet and outlet temperatures, the log mean temperature difference for a *counter-flow* heat exchanger is always *greater* than that for a parallel-flow heat exchanger



$\Delta T_1 = T_{h,\text{in}} - T_{c,\text{out}}$

$\Delta T_2 = T_{h,\text{out}} - T_{c,\text{in}}$

- $\mathbf{\Delta T_{lm,\,CF} > \Delta T_{lm,\,PF}}$ and thus a smaller surface area (and thus a smaller heat exchanger) is needed to achieve a specified heat transfer rate in a counter-flow heat exchanger
- In a counter-flow heat exchanger, the temperature difference between the hot and the cold fluids will remain constant along the heat exchanger when the *heat capacity rates* of the two fluids are *equal*
- ΔT = constant when ($C_h = C_c$ or $\dot{m}_h C_{ph} = \dot{m}_c C_{pc}$). Then we **have** $\mathbf{\Delta T_1 = \Delta T_2}$, and the LMTD relation gives $\mathbf{\Delta T_{lm} = \frac{0}{0}}$, which is indeterminate. It can be shown by the application of L'Hôpital's rule that in this case we have $\mathbf{\Delta T_{lm} = \Delta T_1 = \Delta T_2}$

Multipass and Cross-Flow Heat Exchangers: Use of a Correction Factor

- LMTD relations for *cross-flow* and *multipass shell-and-tube* heat exchangers, are too complicated because of the complex flow conditions. Hence the following relation is used for cross flow and multi pass heat exchangers from the expression of counter flow ΔT_{lm}.

$$\Delta T_{\mathrm{lm}} = F\, \Delta T_{\mathrm{lm},\, CF}$$

- *F* is the **correction factor,** which depends on the *geometry* of the heat exchanger

$\Delta T_{\mathrm{lm},\, CF}$ is the log mean temperature difference for the case of a *counter-flow* heat exchanger with the same inlet and outlet temperatures

$$\Delta T_1 = T_{h,\mathrm{in}} - T_{c,\mathrm{out}}$$
$$\Delta T_2 = T_{h,\mathrm{out}} - T_{c,\mathrm{in}}$$

- The correction factor is less than unity for a cross-flow and multipass shell and-tube heat exchanger, $F \leq 1$

- Correction factor F for a heat exchanger is a measure of deviation of the ΔT_{lm} from the corresponding values for the counter-flow case

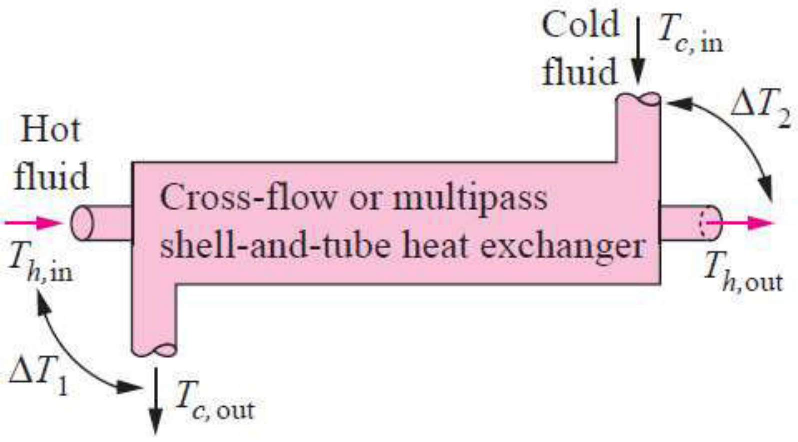
Multipass and Cross-Flow Heat Exchangers: Use of a Correction Factor

Two temperature ratios defined are:

$$P = \frac{t_2 - t_1}{T_1 - t_1} \qquad R = \frac{T_1 - T_2}{t_2 - t_1} = \frac{(\dot{m}C_p)_{\text{tube side}}}{(\dot{m}C_p)_{\text{shell side}}}$$

where the subscripts 1 and 2 represent the inlet and outlet, respectively. For a shell-and-tube heat exchanger, T and t represent the shell- and tube-side temperatures, respectively

P ranges from 0 to 1. The value of *R*, on the other hand, ranges from 0 to infinity, with *R* = 0 corresponding to the phase-change (condensation or boiling) on the shell-side and *R* → ∞ to phase-change on the tube side



Heat transfer rate:

$$\dot{Q} = UA_sF\Delta T_{lm,CF}$$

where

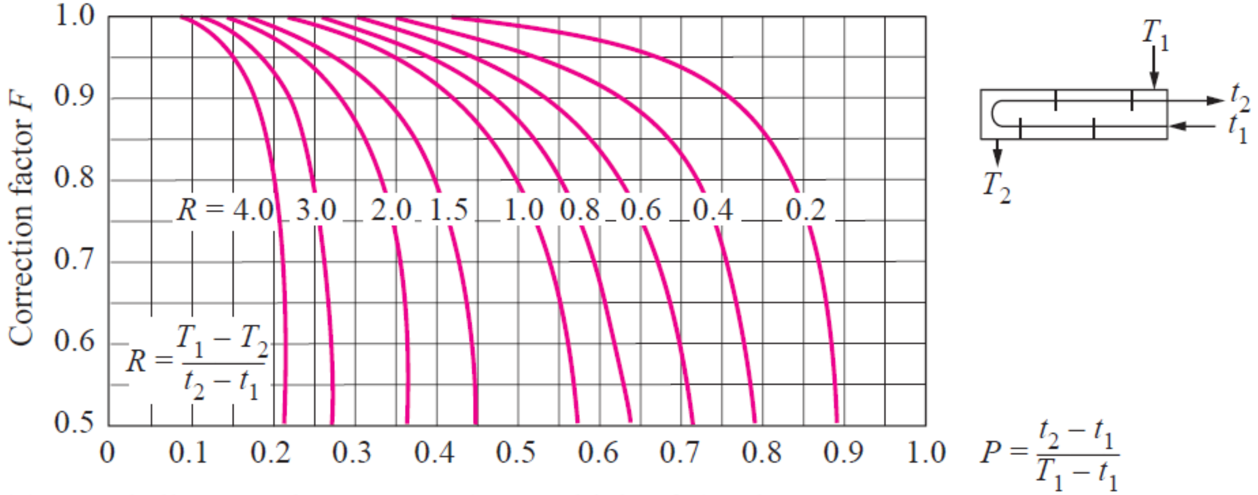
$$\Delta T_{\text{lm},CF} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)}$$

$$\Delta T_1 = T_{h,\text{in}} - T_{c,\text{out}}$$

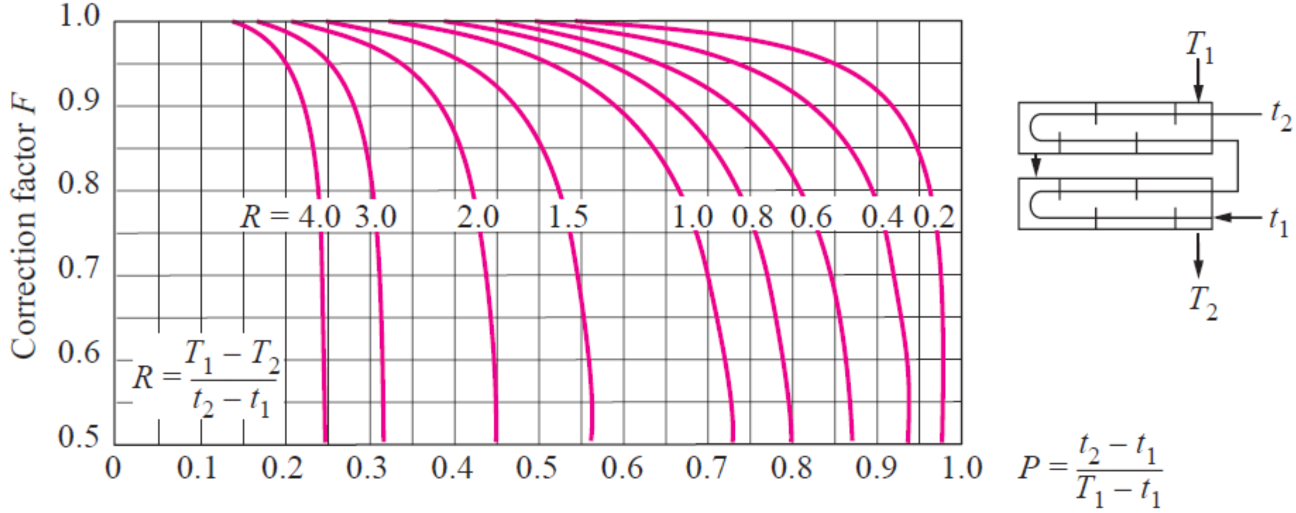
$$\Delta T_2 = T_{h,\text{out}} - T_{c,\text{in}}$$

F is determined from charts

Correction factor *F* charts for common shell-and-tube and cross-flow heat exchangers

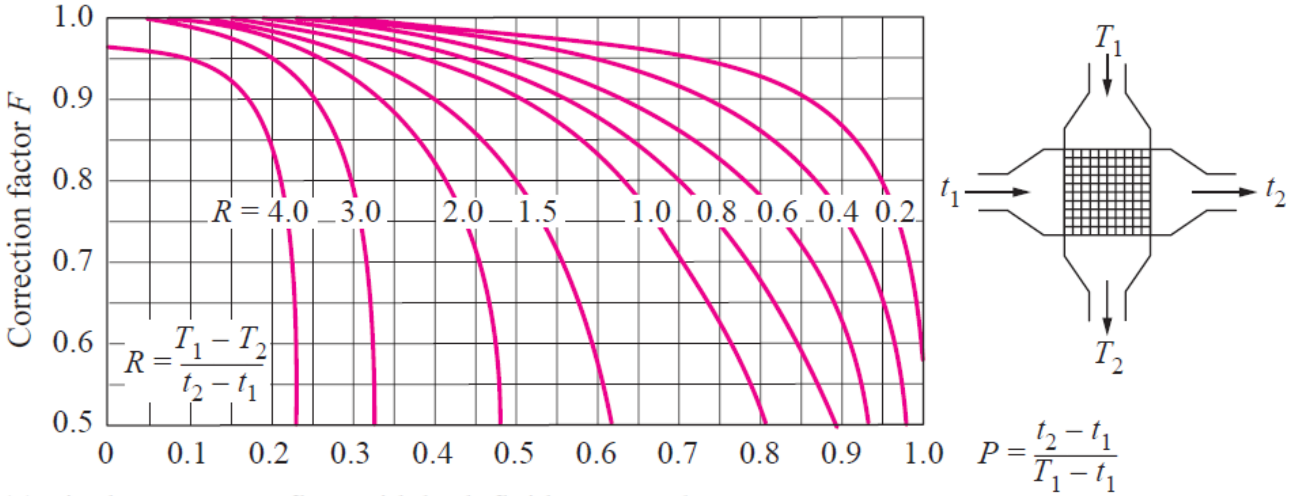


(*a*) One-shell pass and 2, 4, 6, etc. (any multiple of 2), tube passes

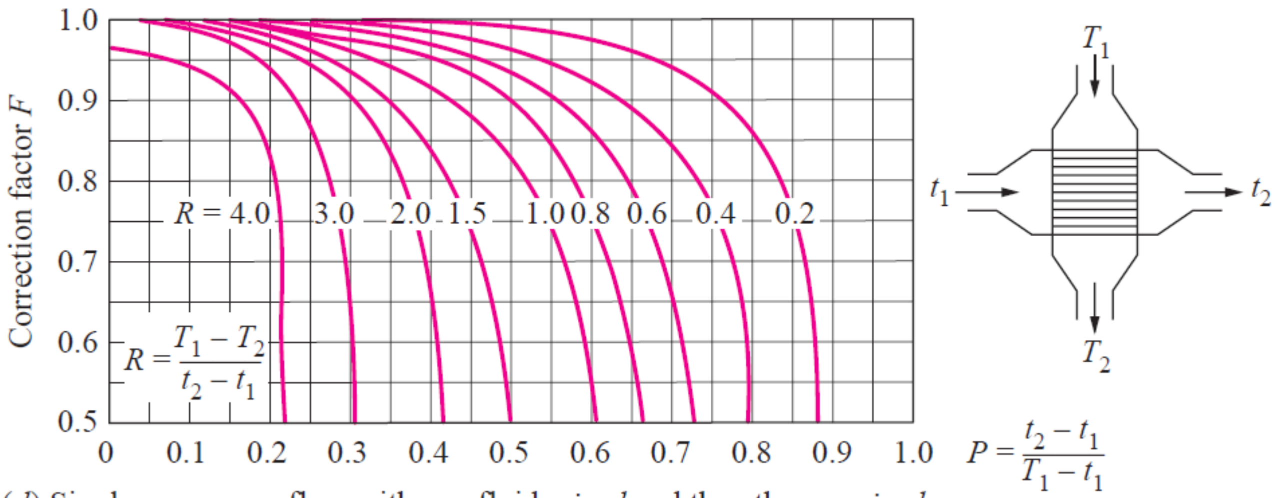


(*b*) Two-shell passes and 4, 8, 12, etc. (any multiple of 4), tube passes

Correction factor *F* charts for common shell-and-tube and cross-flow heat exchangers



(*c*) Single-pass cross-flow with both fluids *unmixed*



(*d*) Single-pass cross-flow with one fluid *mixed* and the other *unmixed*

The effectiveness – NTU method

- LMTD is used in heat exchanger analysis when the inlet and the outlet temperatures of the hot and cold fluids are known or can be determined from an energy balance. This method is very suitable for determining the *size* of a heat exchanger to realize prescribed outlet temperatures when the mass flow rates and the inlet and outlet temperatures of the hot and cold fluids are specified

- A second kind of problem encountered in heat exchanger analysis is the determination of the heat transfer rate and the outlet temperatures of the hot and cold fluids for prescribed fluid mass flow rates and inlet temperatures when the type and size of the heat exchanger are specified.

- The heat transfer surface area A_s of the heat exchanger in the second case is known, but the outlet temperatures are not known. Here the task is to determine the heat transfer performance of a specified heat exchanger or to determine if a heat exchanger available in storage will do the job

- Effectiveness–NTU method, which greatly simplified heat exchanger analysis is based on dimensionless parameter called heat transfer effectiveness ε

$$\varepsilon = \frac{\dot{Q}}{Q_{\max}} = \frac{\text{Actual heat transfer rate}}{\text{Maximum possible heat transfer rate}}$$

The effectiveness – NTU method

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{\text{Actual heat transfer rate}}{\text{Maximum possible heat transfer rate}}$$

- The actual heat transfer rate in a heat exchanger can be determined from an energy balance on the hot or cold fluids and can be expressed as:

$$\dot{Q} = C_c(T_{c,\,\mathrm{out}} - T_{c,\,\mathrm{in}}) = C_h(T_{h,\,\mathrm{in}} - T_{h,\,\mathrm{out}})$$

where C_c and C_h are heat capacities of hot and cold fluids respectively

$$C_c = \dot{m}_c C_{pc} \text{ and } C_h = \dot{m}_c C_{ph}$$

- To determine the maximum possible heat transfer rate in a heat exchanger, we first recognize that the maximum temperature difference in a heat exchanger is the difference between the inlet temperatures of the hot and cold fluids as:

$$\Delta T_{\max} = T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}}$$

- **The heat transfer in a heat exchanger will reach its maximum value when (1) the cold fluid is heated to the inlet temperature of the hot fluid or (2) the hot fluid is cooled to the inlet temperature of the cold fluid. These two limiting conditions will not be reached simultaneously unless the heat capacity rates of the hot and cold fluids are identical (i.e., $C_c = C_h$)**

The effectiveness – NTU method

- **The fluid with smaller heat capacity rate will experience a larger temperature change, and thus it will be the first to experience the maximum temperature, at which point the heat transfer will come to a halt. Therefore, the maximum possible heat transfer rate in a heat exchanger is:**

$$\dot{Q}_{\max} = C_{\min}(T_{h,\,\text{in}} - T_{c,\,\text{in}})$$

where $C_{\min}$ is the smaller of $C_h = \dot{m}_h C_{ph}$ and $C_c = \dot{m}_c C_{pc}$.

- The effectiveness of a heat exchanger enables us to determine the heat transfer rate without knowing the outlet temperatures of the fluids

$$\dot{Q} = \varepsilon \dot{Q}_{\max} = \varepsilon C_{\min}(T_{h,\,\text{in}} - T_{c,\,\text{in}})$$

- The effectiveness of a heat exchanger depends on the geometry of the heat exchanger as well as the flow arrangement

- Different types of heat exchangers have different effectiveness relations

The effectiveness – Parallel flow heat exchanger

During LMTD analysis we derived that:

$$\ln\frac{T_{h,\,\text{out}}-T_{c,\,\text{out}}}{T_{h,\,\text{in}}-T_{c,\,\text{in}}}=-UA_s\left(\frac{1}{\dot{m}_h C_{ph}}+\frac{1}{\dot{m}_c C_{pc}}\right)$$

Since $C_c = \dot{m}_c C_{pc} \text{ and } C_h = \dot{m}_c C_{ph}$

The above LMTD expression can be written as:

$$\ln\frac{T_{h,\,\text{out}}-T_{c,\,\text{out}}}{T_{h,\,\text{in}}-T_{c,\,\text{in}}}=-\frac{UA_s}{C_c}\left(1+\frac{C_c}{C_h}\right)$$ **Eq.1**

However we know from the actual heat transfer expression:

$C_c(T_{c,\,\text{out}}-T_{c,\,\text{in}})=C_h(T_{h,\,\text{in}}-T_{h,\,\text{out}})$ Rearranging this gives below equation

$$T_{h,\,\text{out}}=T_{h,\,\text{in}}-\frac{C_c}{C_h}(T_{c,\,\text{out}}-T_{c,\,\text{in}})$$ **Eq.2**

Adding and subtracting $T_{c,in}$ in L.H.S numerator of Eq.1 and also substituting for $T_{h,out}$ from Eq.2 in Eq.1 gives:

Contd---

The effectiveness – Parallel flow heat exchanger

$$\ln \frac{T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}} + T_{c,\,\mathrm{in}} - T_{c,\,\mathrm{out}} - \dfrac{C_c}{C_h}(T_{c,\,\mathrm{out}} - T_{c,\,\mathrm{in}})}{T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}}} = -\frac{UA_s}{C_c}\left(1 + \frac{C_c}{C_h}\right)$$

which simplifies to:

$$\ln\left[1 - \left(1 + \frac{C_c}{C_h}\right)\frac{T_{c,\,\mathrm{out}} - T_{c,\,\mathrm{in}}}{T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}}}\right] = -\frac{UA_s}{C_c}\left(1 + \frac{C_c}{C_h}\right)$$

Eq.3

From the definition of effectiveness we know that:

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{C_c(T_{c,\,\mathrm{out}} - T_{c,\,\mathrm{in}})}{C_{\min}(T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}})} \quad \Longrightarrow \quad \frac{T_{c,\,\mathrm{out}} - T_{c,\,\mathrm{in}}}{T_{h,\,\mathrm{in}} - T_{c,\,\mathrm{in}}} = \varepsilon\,\frac{C_{\min}}{C_c}$$

Eq.4

Substituting Eq.4 in L.H.S of Eq.3

$$\boxed{\varepsilon_{\text{parallel flow}} = \frac{1 - \exp\left[-\dfrac{UA_s}{C_c}\left(1 + \dfrac{C_c}{C_h}\right)\right]}{\left(1 + \dfrac{C_c}{C_h}\right)\dfrac{C_{\min}}{C_c}}}$$

Contd---

The effectiveness – Parallel flow heat exchanger

Taking either C_c *or* C_h *to be* C_{min} *(both approaches give the same result), the* relation above can be expressed more conveniently as

$$\varepsilon_{\text{parallel flow}} = \frac{1 - \exp\left[-\frac{UA_s}{C_{\min}}\left(1 + \frac{C_{\min}}{C_{\max}}\right)\right]}{1 + \frac{C_{\min}}{C_{\max}}}$$

- C_{min} *is the smaller heat capacity ratio and* C_{max} *is the larger one, and it* makes no difference whether C_{min} *belongs to the hot or cold fluid.*
- *UAs /*C_{min} *is a dimensionless term and is called number of transfer units (NTU)*

$$\text{NTU} = \frac{UA_s}{C_{\min}} = \frac{UA_s}{(\dot{m}C_p)_{\min}}$$

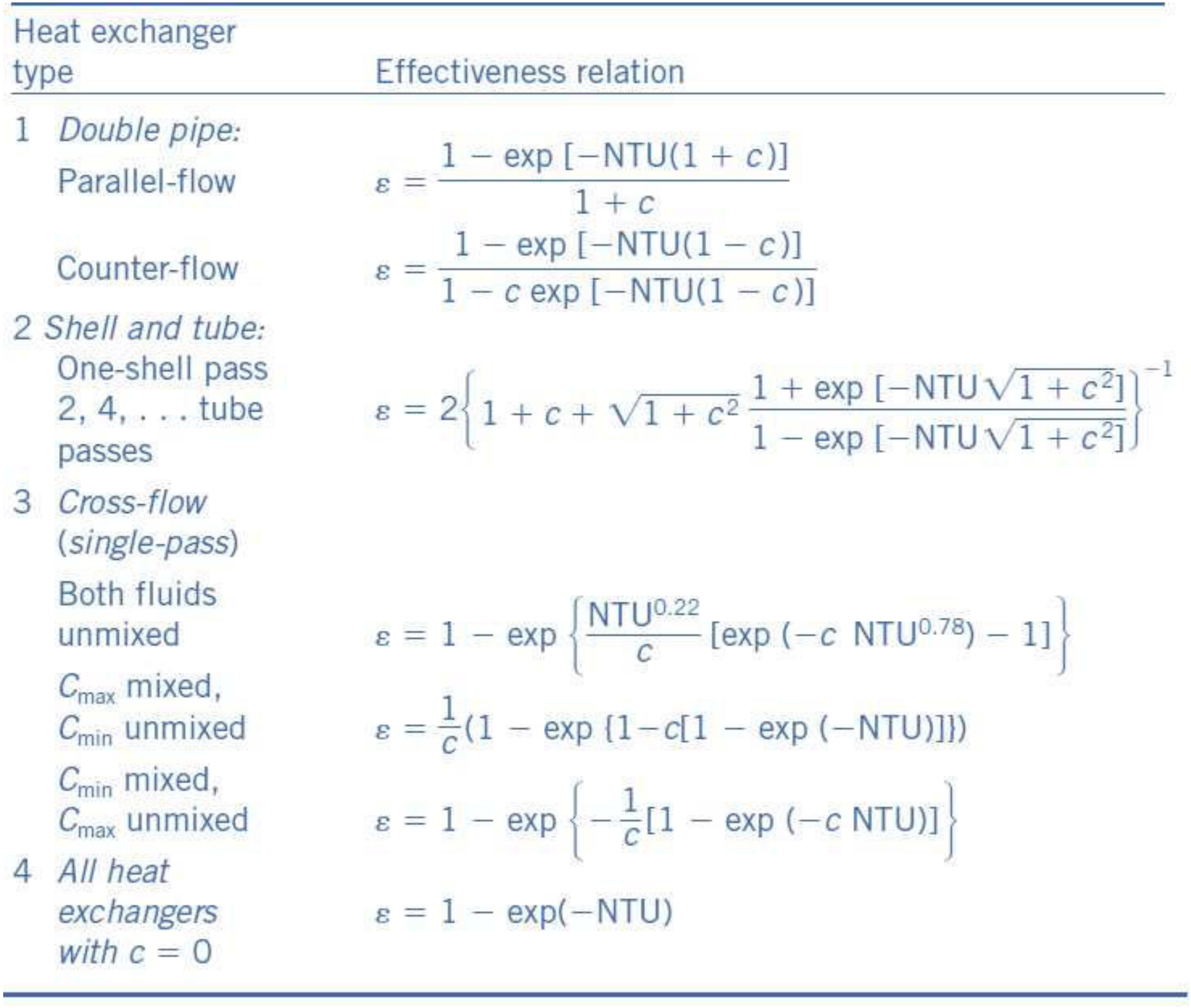
- **For a specified values of** ***U and*** C_{min}***, the value of NTU is a measure of the heat transfer surface area*** A_s***. Thus, the larger the NTU, the larger the heat exchanger***

In heat exchanger analysis, another dimensionless quantity called the ***capacity ratio c as*** ⇨

$$c = \frac{C_{\min}}{C_{\max}}$$

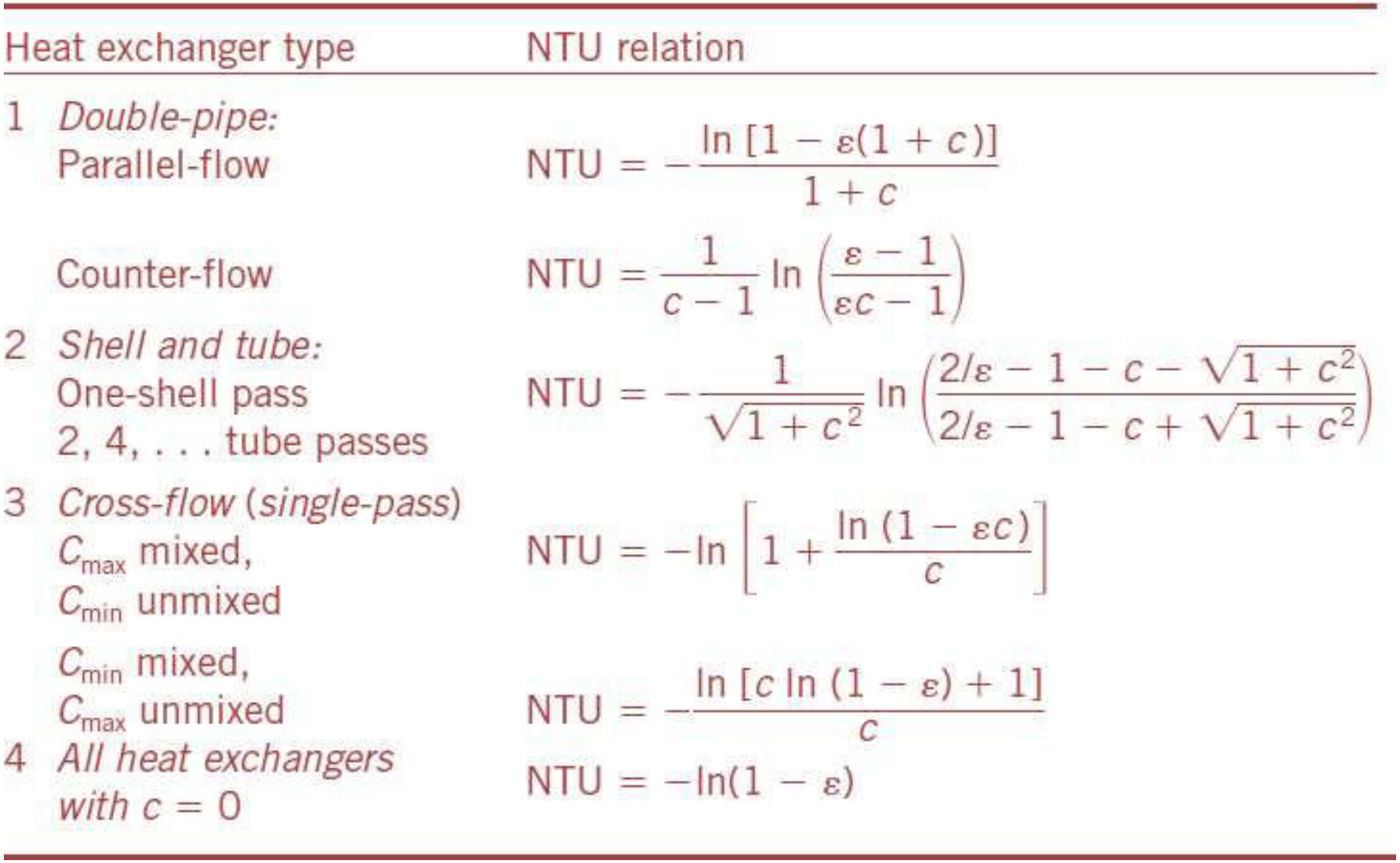
$$\boxed{\varepsilon = \text{function}\,(UA_s/C_{\min}, C_{\min}/C_{\max}) = \text{function}\,(\text{NTU}, c)}$$

Table.I. Effectiveness relation for heat exchangers



Heat exchanger type	Effectiveness relation
1 *Double pipe:*	
Parallel-flow	$\varepsilon = \dfrac{1 - \exp\left[-\mathrm{NTU}(1 + c)\right]}{1 + c}$
Counter-flow	$\varepsilon = \dfrac{1 - \exp\left[-\mathrm{NTU}(1 - c)\right]}{1 - c\exp\left[-\mathrm{NTU}(1 - c)\right]}$
2 *Shell and tube:*	
One-shell pass 2, 4, . . . tube passes	$\varepsilon = 2\left\{1 + c + \sqrt{1 + c^2}\,\dfrac{1 + \exp\left[-\mathrm{NTU}\sqrt{1 + c^2}\right]}{1 - \exp\left[-\mathrm{NTU}\sqrt{1 + c^2}\right]}\right\}^{-1}$
3 *Cross-flow (single-pass)*	
Both fluids unmixed	$\varepsilon = 1 - \exp\left\{\dfrac{\mathrm{NTU}^{0.22}}{c}\left[\exp\left(-c\ \mathrm{NTU}^{0.78}\right) - 1\right]\right\}$
$C_{\max}$ mixed, $C_{\min}$ unmixed	$\varepsilon = \dfrac{1}{c}\left(1 - \exp\left\{1 - c\left[1 - \exp\left(-\mathrm{NTU}\right)\right]\right\}\right)$
$C_{\min}$ mixed, $C_{\max}$ unmixed	$\varepsilon = 1 - \exp\left\{-\dfrac{1}{c}\left[1 - \exp\left(-c\ \mathrm{NTU}\right)\right]\right\}$
4 *All heat exchangers with* $c = 0$	$\varepsilon = 1 - \exp(-\mathrm{NTU})$

Table.II. Effectiveness relation for heat exchangers



Heat exchanger type	NTU relation
1 *Double-pipe:* Parallel-flow	$\text{NTU} = -\dfrac{\ln[1-\varepsilon(1+c)]}{1+c}$
Counter-flow	$\text{NTU} = \dfrac{1}{c-1}\ln\left(\dfrac{\varepsilon-1}{\varepsilon c-1}\right)$
2 *Shell and tube:* One-shell pass 2, 4, . . . tube passes	$\text{NTU} = -\dfrac{1}{\sqrt{1+c^2}}\ln\left(\dfrac{2/\varepsilon-1-c-\sqrt{1+c^2}}{2/\varepsilon-1-c+\sqrt{1+c^2}}\right)$
3 *Cross-flow (single-pass)* C_{max} mixed, C_{min} unmixed	$\text{NTU} = -\ln\left[1+\dfrac{\ln(1-\varepsilon c)}{c}\right]$
C_{min} mixed, C_{max} unmixed	$\text{NTU} = -\dfrac{\ln[c\ln(1-\varepsilon)+1]}{c}$
4 *All heat exchangers* *with $c = 0$*	$\text{NTU} = -\ln(1-\varepsilon)$

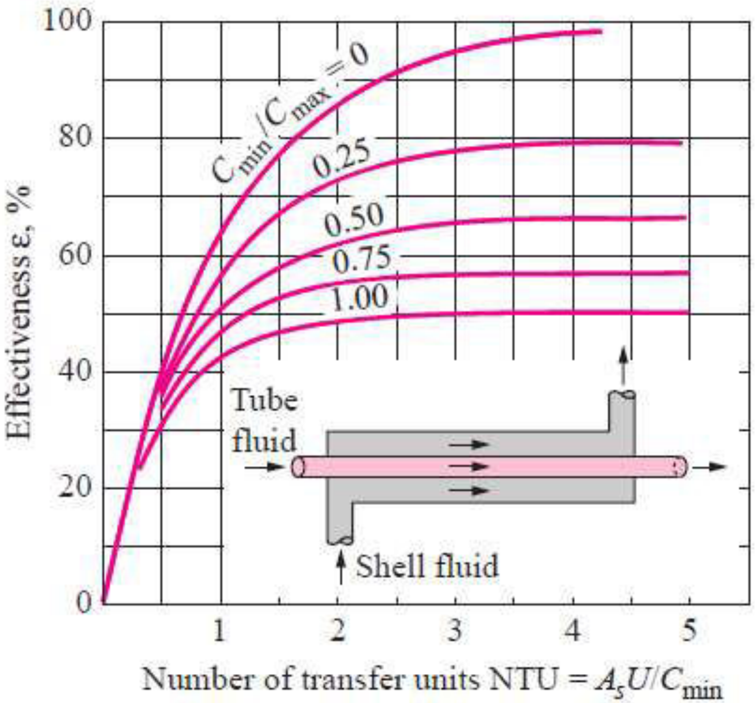
The effectiveness relations given in Table. I and Table. II are equivalent. Table.I gives the effectiveness (ε) directly when NTU is known and the numerical relations in Table.II give NTU directly when ε is known

General observations on effectiveness relations

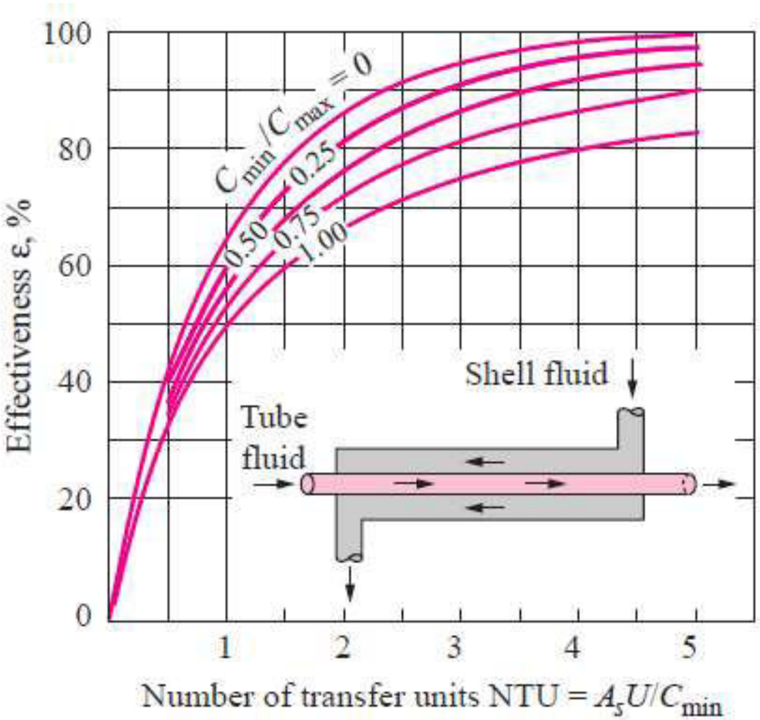
- The value of the effectiveness ranges from 0 to 1. **It increases rapidly with NTU for small values (up to about NTU 1.5)** but rather slowly for larger values

- Heat exchanger with a large NTU (usually larger than 3) and thus a large size cannot be justified economically, since a large increase in NTU in this case corresponds to a small increase in effectiveness

- Thus, a heat exchanger with a very high effectiveness may be highly desirable from a heat transfer point of view but rather undesirable from an economical point of view



(*a*) Parallel-flow



(*b*) Counter-flow

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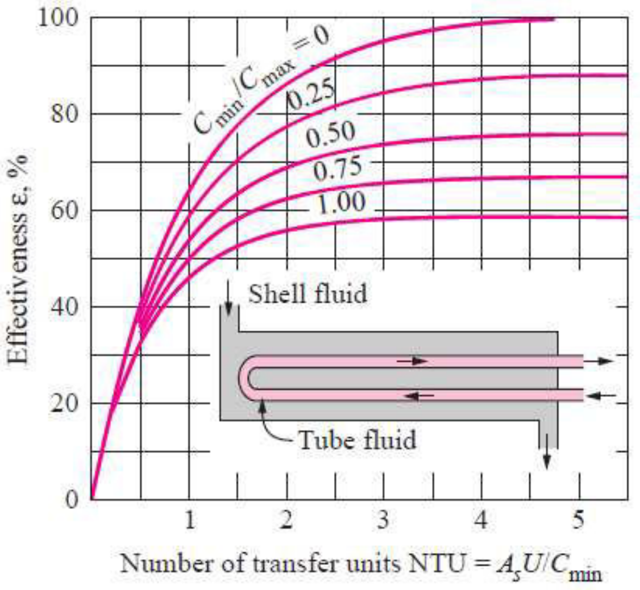
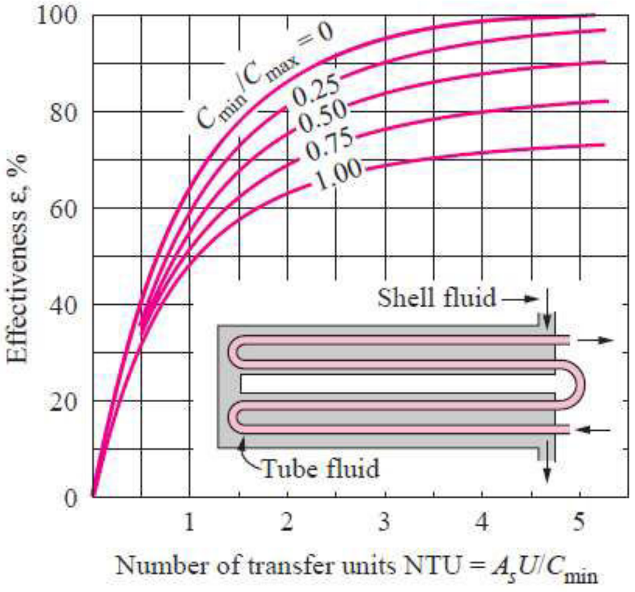
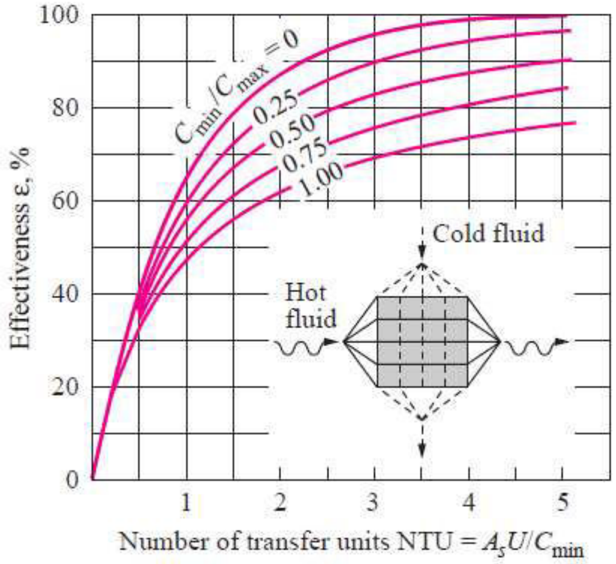


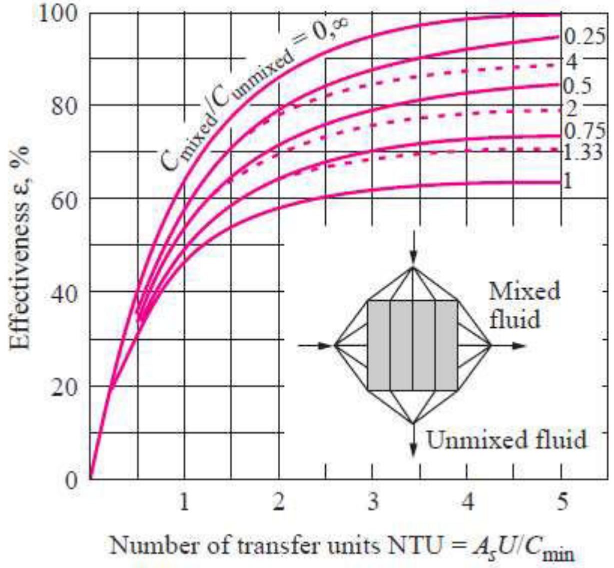
(*c*) One-shell pass and 2, 4, 6, … tube passes



(*d*) Two-shell passes and 4, 8, 12, … tube passes



(*e*) Cross-flow with both fluids unmixed

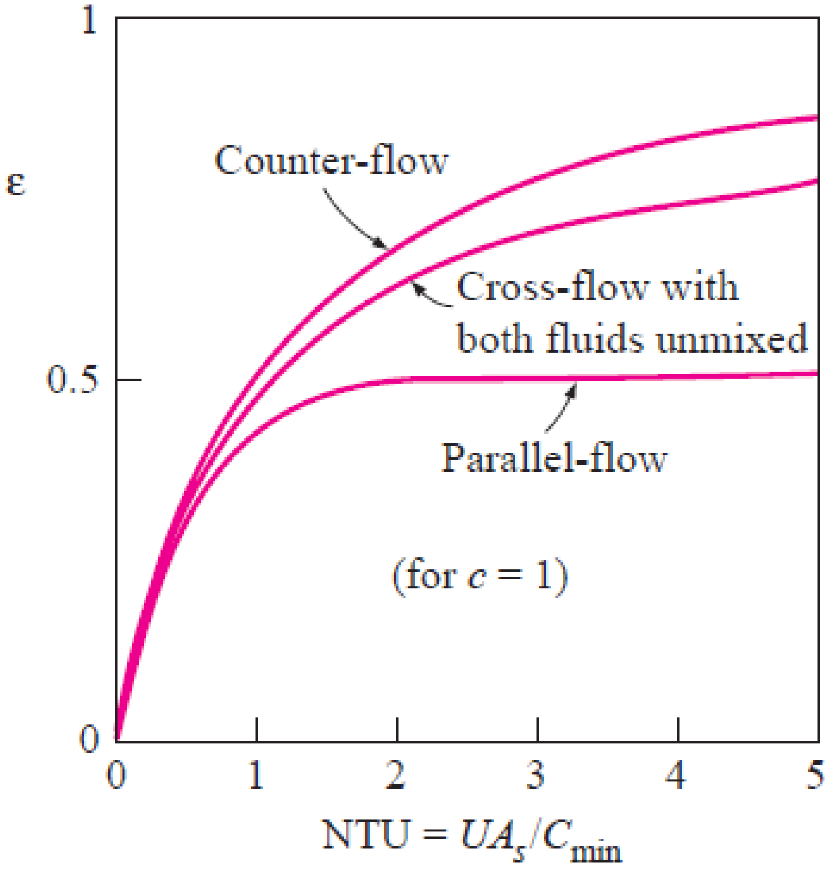


(*f*) Cross-flow with one fluid mixed and the other unmixed

General observations on effectiveness relations

- For a given NTU and capacity ratio *c* = C_{min} /C_{max}, *the counter-flow* heat exchanger has the *highest effectiveness, followed closely by the* cross-flow heat exchangers with both fluids unmixed. The lowest effectiveness values are encountered in parallel-flow heat exchangers

- The effectiveness of a heat exchanger is independent of the capacity ratio, *c for NTU values of less than about 0.3*



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General observations on effectiveness relations

- The value of the capacity ratio, *c ranges between 0 and 1. For a given* NTU, the effectiveness becomes a *maximum for c = 0 and a minimum* for *c = 1. The case c* $C_{min}/C_{max} \rightarrow 0$ *corresponds to* $C_{max} \rightarrow \infty$ *, which* is realized during a phase-change process in a *condenser or boiler. All* effectiveness relations in this case reduce to:

$$\varepsilon = \varepsilon_{\max} = 1 - \exp(-\mathrm{NTU})$$